Goodness-of-Fit Testing using Wasserstein Distance to evaluate model of Double Electron Capture background

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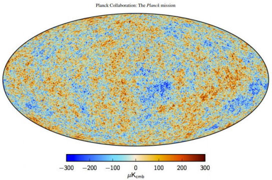
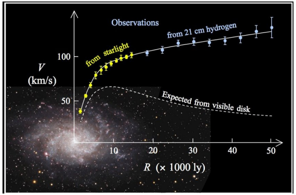
Introduction to XENON

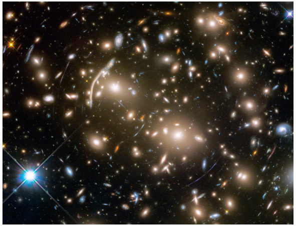
Dark Matter and Other Exotic Particles



Temperature Fluctuations in CMB [7]



Rotation Curves of Speeds of Galaxies [9]



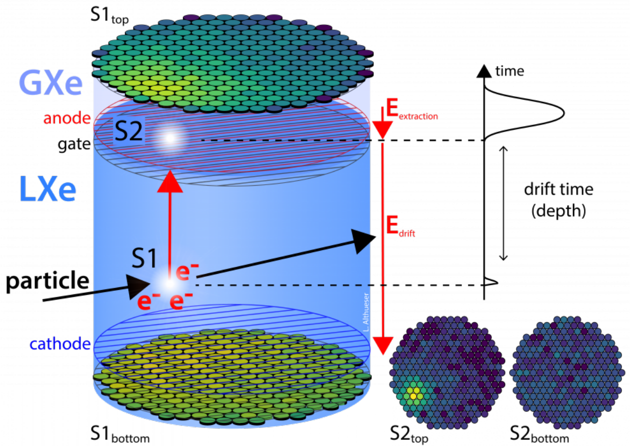
Gravitational Lensing [10]

XENON Collaboration

XENON Detector [4]

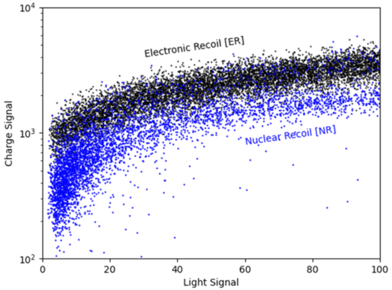
Laboratori Nazionali del Gran Sasso (LNGS) [8]

Dual Phase Xenon Time Projection Chamber (TPC)



[1]

ER and NR Discrimination



[12]

ER:

- ◇ Electronic Recoil
- ◇ Particle collides with electron
- ◇ Background
- ◇ low ionization density → low recombination rate → higher charge yield

NR:

- ◇ Nuclear Recoil
- ◇ Particle collides with nucleus / Xe atom
- ◇ ROI to find WIMPs
- ◇ high ionization density → high recombination rate → lower charge yield

Double Electron Capture (DEC)

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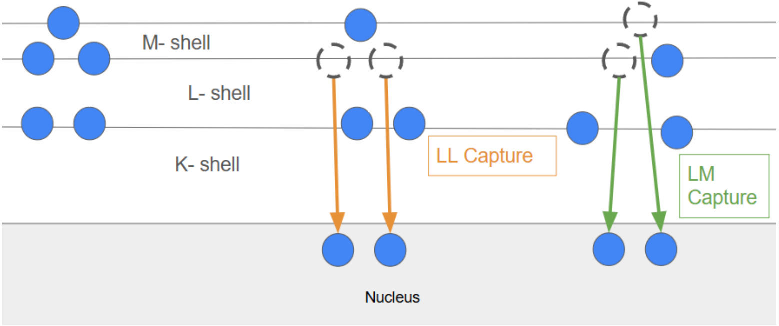
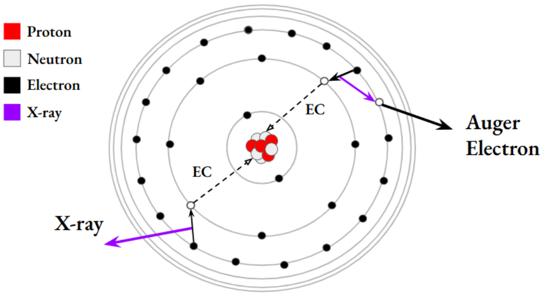
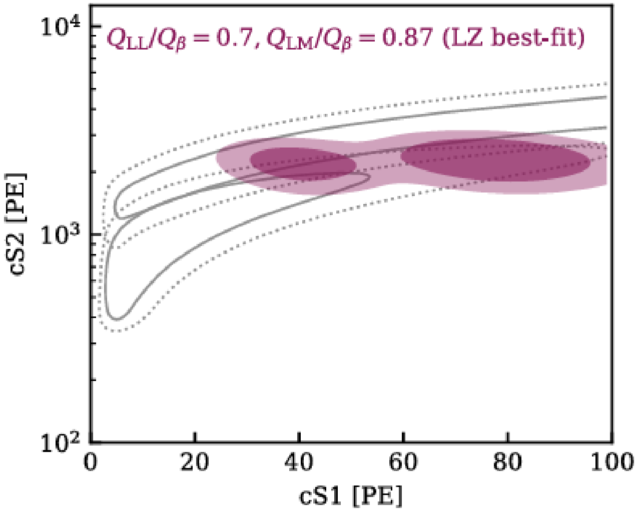


Double Electron Capture (DEC)

$$^{124}\mathrm{Xe} + 2e^- \rightarrow ^{124}\mathrm{Te} + 2\nu_e + \text{(X-rays and Auger electrons)}$$



Double Electron Capture (DEC)



[6]

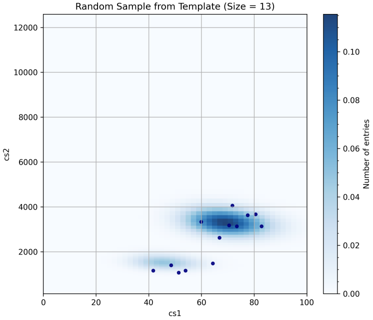
Hypothesis Testing

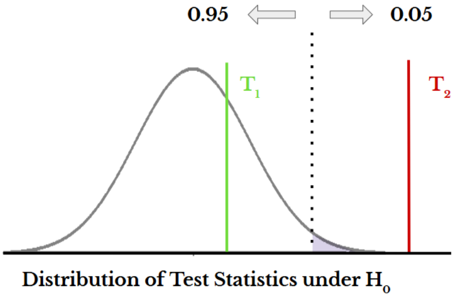
Hypothesis Testing

Goodness of Fit Test - One Tail

Goodness of Fit Test

- Calculate the test statistic - the difference between data and model.
- Compare statistic to test distribution under H_0 to see if it falls beyond the defined confidence interval (e.g. 95%).
- If beyond, reject the null.



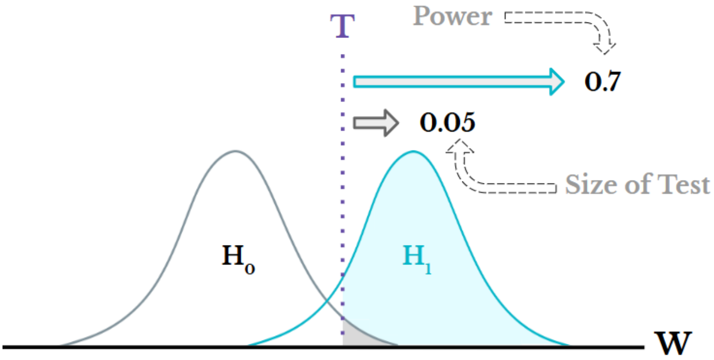


Hypothesis Testing

Power

Power Calculation

$$\text{Power} = \mathrm{P}_{H_1}(W > T)$$



Wasserstein Distance

Wasserstein Distance

General Definition

General Definition: Wasserstein Distance

The Wasserstein distance, also known as the Earth Mover's Distance (EMD), measures the minimum "work" required to transform one probability distribution into another, where "work" = (amount of mass moved) × (distance transported).

Formal Definition:

$$W_p(\mu,\nu) = \left(\inf_{\gamma \in \Gamma(\mu,\nu)} \int_{\mathbb{R}^d \times \mathbb{R}^d} \|x-y\|^p \, d\gamma(x,y) \right)^{1/p}$$

Where:

- μ, ν: input distributions
- $\gamma \in \Gamma(\mu,\nu)$: set of all joint couplings with marginals μ, ν
- $d(x,y)$: metric distance between points x and y

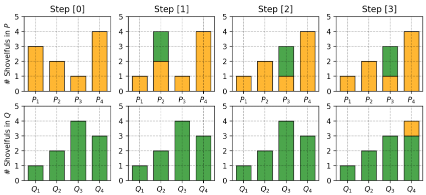


Figure 1. Discrete 1D example of EMD [11].

Wasserstein Distance

EMD Function

2-Wasserstein Distance (EMD) Formulation

Source: $\{x_i\}_{i=1}^{n} \subset \mathbb{R}^2$, $\mathbf{a} \in \mathbb{R}^n$
Target: $\{y_j\}_{j=1}^{m} \subset \mathbb{R}^2$, $\mathbf{b} \in \mathbb{R}^m$

$$\sum_{i=1}^{n} a_i = 1, \quad \sum_{j=1}^{m} b_j = 1, \quad a_i, b_j \geq 0$$

Cost matrix: $C_{ij} = \|x_i - y_j\|_2^2$

Transport plan: $\gamma \in \mathbb{R}^{n \times m}$

$$\sum_{j=1}^{m} \gamma_{ij} = a_i \quad \forall i \qquad \gamma_{ij} \geq 0 \quad \forall i, j$$
$$\sum_{i=1}^{n} \gamma_{ij} = b_j \quad \forall j$$

2-Wasserstein Distance:

$$W = \left(\sum_{i=1}^{n} \sum_{j=1}^{m} \gamma_{ij} C_{ij} \right)^{1/2}$$

Numerical Example: Sample Initialization

Source Distribution

Event Locations: $x = \left\{ \begin{matrix} (1,2), & (1,2), \\ (3,4), & (3,4) \end{matrix} \right\}$

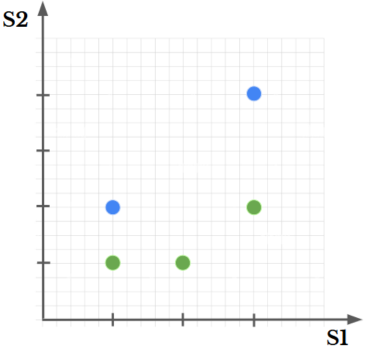
Locations: $x = \{(1,2),\ (3,4)\}$

Masses: $\mathbf{a} = [0.5,\ 0.5]$

Target Distribution

Locations: $y = \{(2,1),\ (3,2),\ (1,1)\}$

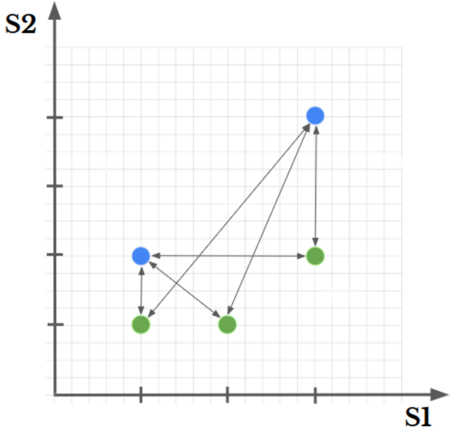
Masses: $\mathbf{b} = [0.25,\ 0.5,\ 0.25]$



Numerical Example: Cost Matrix $C_{ij} = \|x_i - y_j\|_2^2$

	$y_1 = (2,1)$	$y_2 = (3,2)$	$y_3 = (1,1)$
$x_1 = (1,2)$	2	4	1
$x_2 = (3,4)$	10	4	13

$$C = \begin{bmatrix} 2 & 4 & 1 \\ 10 & 4 & 13 \end{bmatrix}$$



Mass Marginals and Optimal Transport Plan

Mass marginals:

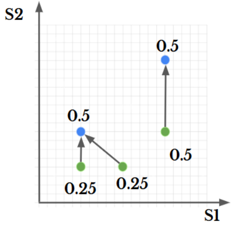
$$\mathbf{a} = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 0.25 \\ 0.5 \\ 0.25 \end{bmatrix}$$

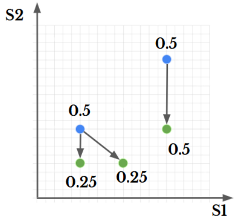
$$\sum_{i=1}^{n} a_i = 1, \quad \sum_{j=1}^{m} b_j = 1, \quad a_i, b_j \geq 0$$

Optimal transport plan:

$$\gamma = \begin{bmatrix} \gamma_{11} & \gamma_{12} & \gamma_{13} \\ \gamma_{21} & \gamma_{22} & \gamma_{23} \end{bmatrix}$$

$$\begin{aligned} &\textstyle\sum_{j=1}^{3} \gamma_{ij} = a_i && \text{(row sums)} \\ &\textstyle\sum_{i=1}^{2} \gamma_{ij} = b_j && \text{(column sums)} \\ &\gamma_{ij} \geq 0 && \text{(non-negativity)} \end{aligned}$$





Solution: $\gamma = \begin{bmatrix} 0.25 & 0 & 0.25 \\ 0 & 0.5 & 0 \end{bmatrix}$

Computing the 2-Wasserstein Distance

$$C = \begin{bmatrix} 2 & 4 & 1 \\ 10 & 4 & 13 \end{bmatrix}, \quad \gamma = \begin{bmatrix} 0.25 & 0 & 0.25 \\ 0 & 0.5 & 0 \end{bmatrix}$$

Compute total cost:

$$\sum_{i=1}^{2} \sum_{j=1}^{2} \gamma_{ij} C_{ij} = 0.25 \cdot 2 + 0.25 \cdot 1 + 0.5 \cdot 4 = 2.75$$

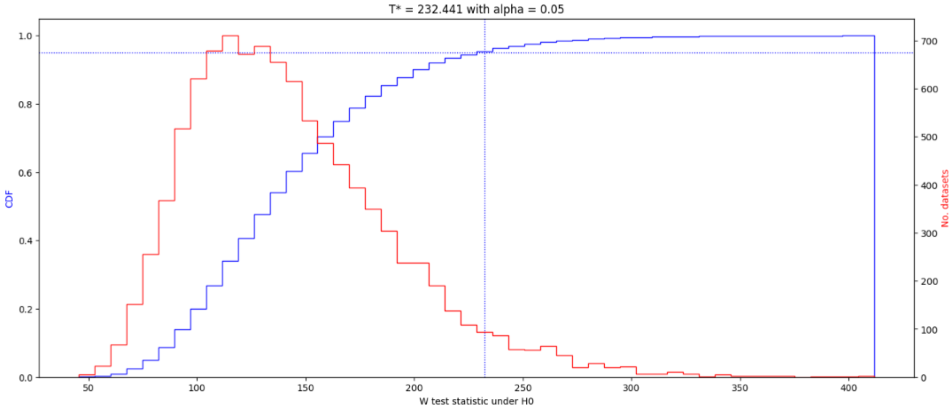
2-Wasserstein Distance:

$$W = \left(\sum_{i,j} \gamma_{ij} C_{ij} \right)^{1/2} = \sqrt{2.75} \approx 1.66$$

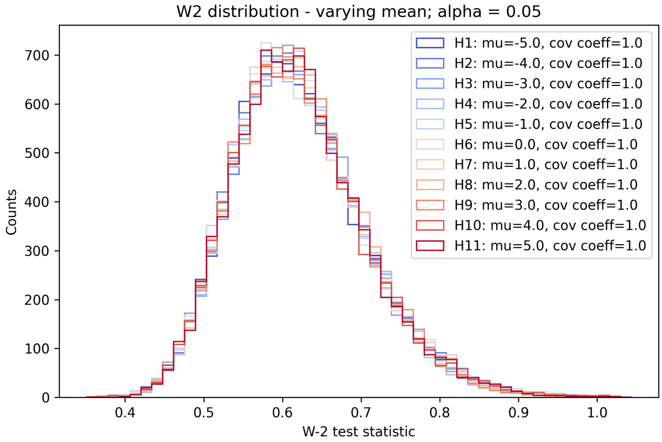
Wasserstein Distance

Wasserstein Distribution

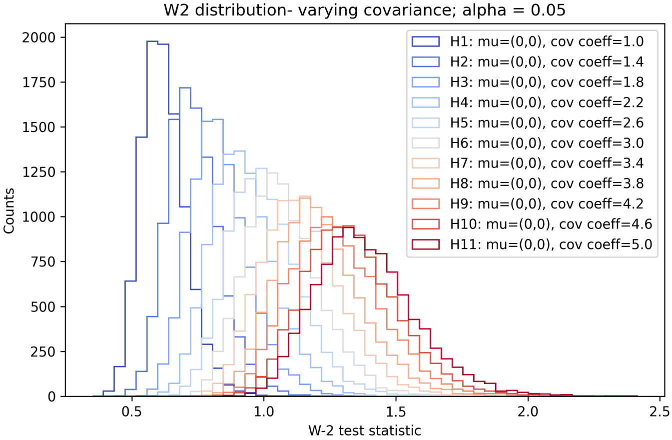
Visualizing a Wasserstein Distribution



Distribution Free? - Varying Mean



Distribution Free? - Varying Covariance



Conclusion

- Not Distribution Free
- Monte Carlo Random Sampling to approximate distribution for each H_0

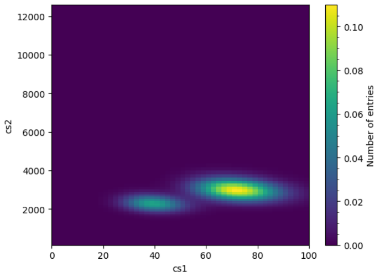
Methodology

Methodology

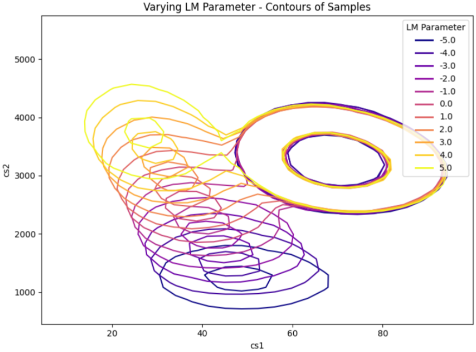
Templates

Template

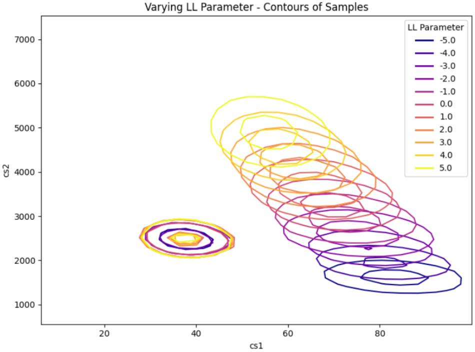
- 2D histogram
- vary LL and LM parameter by changing the light and charge yield
- 11 variations of LL Parameter/11 variations of LM Parameter - 121 possible templates



Vary LL and LM Parameter Individually



Varying LM Parameter - LL Parameter fixed at 0.0



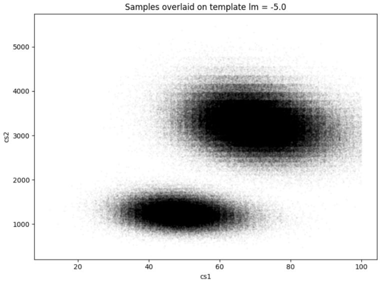
Varying LL Parameter - LM Parameter fixed at 0.0

Methodology

Sampling

Samples

- Sample size = 13 $\rightarrow$ expect to see approximately 13 DEC Events in the data set
- Template - use bin centers and "height" of bin to construct probability distribution
- Redraw sample and compare with template 10,000 times to create Wasserstein Distribution

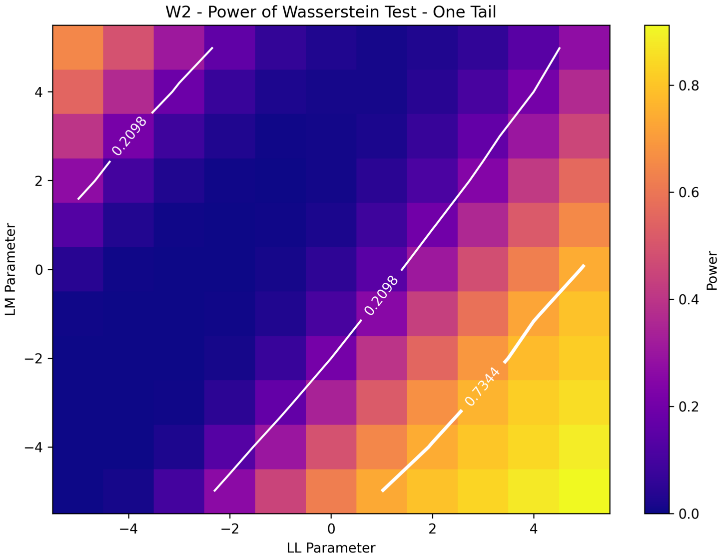


Conclusion

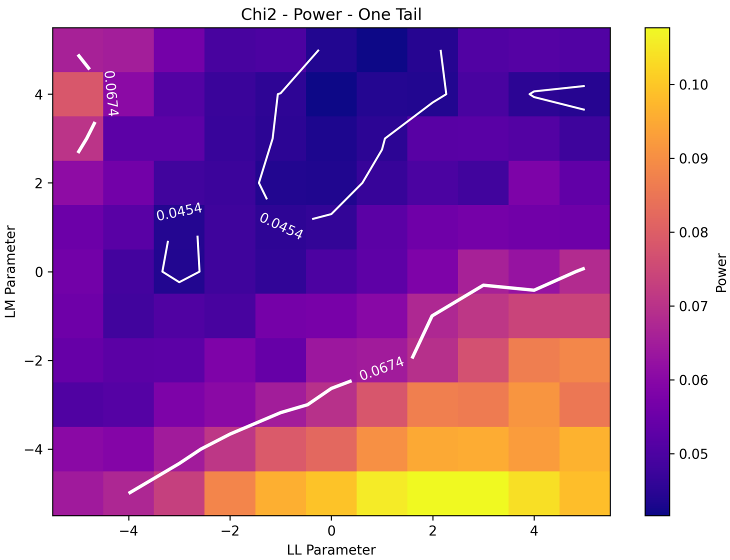
Conclusion

Results

Power Heatmap



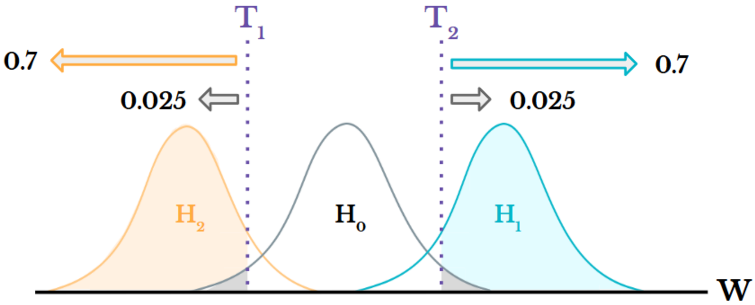
χ^2 Test Comparison



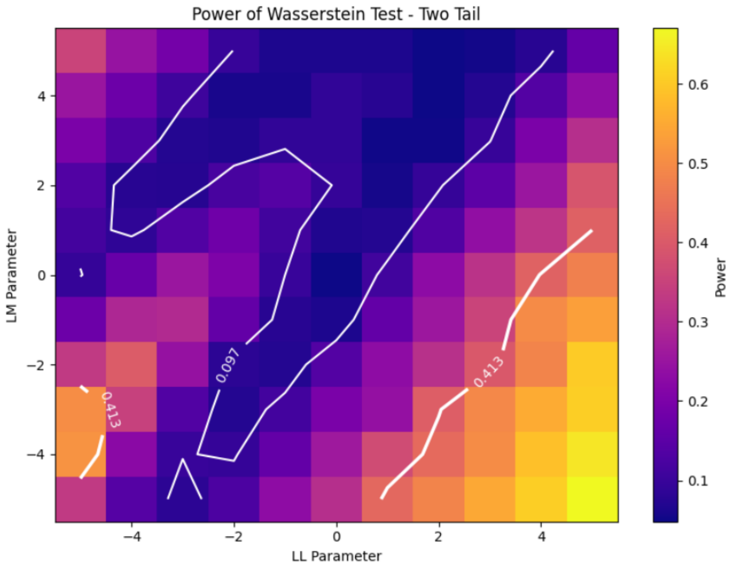
Conclusion

Future Work

Two-Tail Power



Two-Tail Power Heatmap



Thank you!

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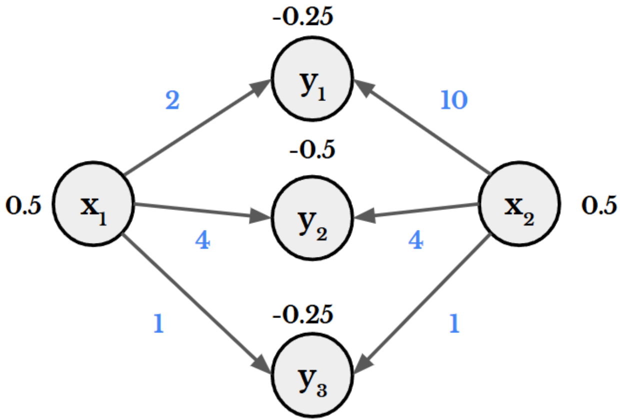
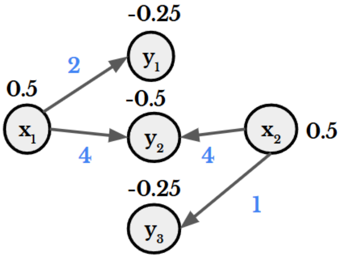
Backup



Backup

Choose Basis:

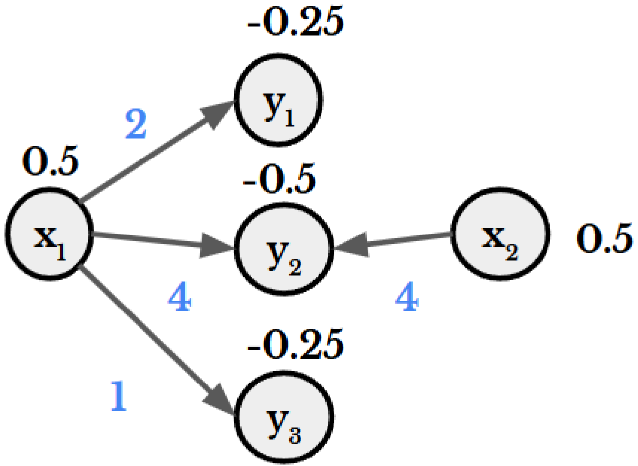


Cost - Dual Tail + Dual Head = 0

2 - 0 - 2 = 0
4 - 0 - 4 = 0
4 - 0 - 4 = 0
1 - 0 - 1 = 0

x_2y_1: 10 - 0 - 2 = 8
x_1y_3: 1 - 0 - 4 = -3

Backup



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