Special Problem: Compact Method "Like Example 2"

Use right-left symmetry of the function f(u), for now in dimensionless form, with f(u)=4u/π in interval 0, π/4; f(u)=4(1-u/π) in interval 3π/4, π and f(u)=1 from π/4 to 3π/4

Use Fourier Sine Series $f(u)=\sum_1^\infty b_n \sin nu,$ **with** $b_n=\frac{2}{\pi}\int_0^\pi f(u)\sin nudu$

We note f(u) has even reflection symmetry around u=π/2, so b_n=0 for n even (0,2,4,...) and $b_n=\frac{4}{\pi}\int_0^{\pi/2} f(u)\sin nudu$ **for odd n**

$$b_n=\frac{4}{\pi}\int_0^{\pi/2}\frac{4}{\pi}u\sin nudu+\frac{4}{\pi}\int_{\pi/4}^{\pi/2}\sin nudu=\frac{16}{\pi}\left[\frac{\sin nu}{n^2}-\frac{u}{n}\cos nu\right]_0^{\frac{\pi}{4}}+\frac{4}{\pi}\left[-\frac{\cos nu}{n}\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$ **for n = 1,3,...**

$$b_n=\frac{16}{\pi}\left[\frac{\sin n\pi/4}{n^2}-\frac{\sin n0}{n^2}-\frac{\pi}{4}\frac{\cos n\pi/4}{n}+\frac{0\cos 0}{n}\right]+\frac{4}{\pi}\left[\frac{-\cos n\pi/2}{n}+\frac{\cos n\pi/4}{n}\right]$$

and we note that the cos terms cancel or are zero, so

$b_n=\frac{16}{\pi^2}\frac{\sin n\pi/4}{n^2}$ **and the sines are all** $1/\sqrt{2}$**, with a pattern of signs(+,+,-,-,+,+,-,-...)**

so $b_1=\frac{8\sqrt{2}}{\pi^2}=1.143, b_3=\frac{8\sqrt{2}}{\pi^2}\frac{1}{9}=0.127, b_5=-\frac{8\sqrt{2}}{\pi^2}\frac{1}{25}=-0.046, b_7=-\frac{8\sqrt{2}}{\pi^2}\frac{1}{49}=-0.023,$

$b_9=\frac{8\sqrt{2}}{\pi^2}\frac{1}{81}=0.014, b_{11}=\frac{8\sqrt{2}}{\pi^2}\frac{1}{121}=0.009, b_{13}=-\frac{8\sqrt{2}}{\pi^2}\frac{1}{169}=-0.007, b_{15}=-\frac{8\sqrt{2}}{\pi^2}\frac{1}{225}=-0.005$

and the actual amplitudes are these numbers times 0.005m, then we compute the energy in a piece dx,

$dE=\frac{1}{2}a^2\omega^2 dm=\frac{1}{2}a^2\omega^2\mu dx$ **and** $a=0.005\sin k_n x b_n$ **for mode n at point x. Then E** $=\int_0^L dE$

$E_n=\frac{1}{2}(0.005b_n\omega_n)^2\mu\int_0^L\sin^2(n\pi\frac{x}{L})dx=\frac{1}{4}(0.005b_n\omega_n)^2\mu L$ **and we know** $\omega_n=\frac{2\pi\nu}{\lambda}=\frac{2\pi}{\lambda}\sqrt{\frac{T}{\mu}}$

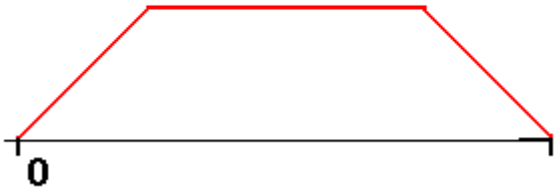
$\lambda=2L/n,\ \omega_n=\frac{n\pi}{L}\sqrt{\frac{T}{\mu}}=\frac{n\pi}{4}\sqrt{\frac{100}{0.020}}=55.5n$ **per s,** $E_n=\frac{(.005\times 55.5nb_n)^2\times.08}{4}=0.00154n^2b_n^2$ **J**

then $\mathbf{E}_1=2.01mJ, E_3=0.224mJ, E_5=0.081mJ, E_7=0.040mJ, E_9=0.024mJ, E_{11}=0.015mJ$

CHECK b_n $f(\pi/2)=1.146(1-1/9-1/25+1/49+1/81-1/121)=0.99$

should =1 so pretty good

Expanded solution, with pictures and comments.
Special Problem: set up "Like Example 2"
Use right-left symmetry of the function f(u). Since the arguments of the sines and cosines always have to be angles, we might as well work in dimensionless form, that is with a variable u ranging from – pi to +pi, and with amplitude equal to 1. We will put in dimensions at the end, as you will see. for now in dimensionless form.
We take f(u)=4u/π in interval 0, π/4; f(u)=4(1-u/π) in interval 3π/4, π and f(u)=1 from π/4 to 3π/4



pi (at 4m)

Use Fourier Sine Series $f(u)=\sum_1^\infty b_n \sin nu,\ \text{with} b_n = \frac{2}{\pi}\int_0^\pi f(u)\sin nu du$

I have evaluated that integral, and find that a lot of terms cancel. To save writing, I chose to take advantage of the symmetry of the above function about its midpoint, and integrate over the left-hand half, getting the answer in a more compact form. Here is the story for that method.
We note f(u) has even reflection symmetry around u=π/2, This means that f(pi/2 – s) = f(pi/2 + s) for s less that pi/2. This we call an even function around pi/2. Sin(nu) is an even function around pi/2 also, if n is odd, 1,3,5.. so as we see from the picture above, with a sin(u) superimposed in green, the integral for odd n is the same for integrating from 0 to pi/2, and pi/2 to pi, so I

can just do the 0 to pi/2 and double the result to evaluate the whole. (sin (nu) for n even is an odd function around pi/2, so the integrals in the two intervals cancel. Indeed, when I did the full integral over zero to pi, the value of the integral and so the b_n for n even, is zero.)

so b_n=0 for n even (0,2,4,...) and $b_n = \frac{4}{\pi}\int_0^{\pi/2} f(u)\sin nu\,du$ **for odd n**

$$b_n = \frac{4}{\pi}\int_0^{\pi/2}\frac{4}{\pi}u\sin nu\,du + \frac{4}{\pi}\int_{\pi/4}^{\pi/2}\sin nu\,du = \frac{16}{\pi}\left[\frac{\sin nu}{n^2} - \frac{u}{n}\cos nu\right]_0^{\frac{\pi}{4}} + \frac{4}{\pi}\left[-\frac{\cos nu}{n}\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} \quad \textbf{for n = 1,3,...}$$

$$b_n = \frac{16}{\pi^2}\left[\frac{\sin n\pi/4}{n^2} - \frac{\sin n0}{n^2} - \frac{\pi}{4}\frac{\cos n\pi/4}{n} + \frac{0\cos 0}{n}\right] + \frac{4}{\pi}\left[\frac{-\cos n\pi/2}{n} + \frac{\cos n\pi/4}{n}\right]$$

and we note that the cos terms cancel or are zero, so

$b_n = \frac{16}{\pi^2}\frac{\sin n\pi/4}{n^2}$ **and the sines are all** $1/\sqrt{2}$**, with a pattern of signs(+,+,-,-,+,+,-,-...)**

so $b_1 = \frac{8\sqrt{2}}{\pi^2} = 1.143, b_3 = \frac{8\sqrt{2}}{\pi^2}\frac{1}{9} = 0.127, b_5 = -\frac{8\sqrt{2}}{\pi^2}\frac{1}{25} = -0.046, b_7 = -\frac{8\sqrt{2}}{\pi^2}\frac{1}{49} = -0.023,$

$b_9 = \frac{8\sqrt{2}}{\pi^2}\frac{1}{81} = 0.014, b_{11} = \frac{8\sqrt{2}}{\pi^2}\frac{1}{121} = 0.009, b_{13} = -\frac{8\sqrt{2}}{\pi^2}\frac{1}{169} = -0.007, b_{15} = -\frac{8\sqrt{2}}{\pi^2}\frac{1}{225} = -0.005$

and the actual amplitudes are these numbers times 0.005m. Let us check to see if these coefficients are right: CHECK b_n $f(\pi/2) = 1.146(1 - 1/9 - 1/25 + 1/49 + 1/81 - 1/121) = 0.99$
since this should =1 the result of the check is pretty good.

We can apply this Fourier Series for the shape at time zero either to a standing wave, with the ends of the string fixed, or to a traveling wave which has this pulse shape moving along the wire. I will go through both cases to make the connection between the two.

For the standing wave, our coefficients give the solution

$$y(x,t) = 0.005\sum_n b_n \cos(n\frac{\pi}{L}x)\cos\omega_n t$$

For the standing wave, we compute the energy in a piece dx,

$dE = \frac{1}{2}a^2\omega^2 dm = \frac{1}{2}a^2\omega^2\mu dx$ **and** $a = 0.005(\sin\frac{n\pi}{L}x)b_n$ **for mode n at point x. Then E** $= \int_0^L dE$

$E_n = \frac{1}{2}(0.005b_n\omega_n)^2\mu\int_0^L \sin^2(n\pi\frac{x}{L})dx = \frac{1}{4}(0.005b_n\omega_n)^2\mu L$ **and we know** $\omega_n = \frac{2\pi v}{\lambda_n} = \frac{2\pi}{\lambda_n}\sqrt{\frac{T}{\mu}}$

$\lambda = 2L/n,\ \omega_n = \frac{n\pi}{L}\sqrt{\frac{T}{\mu}} = \frac{n\pi}{4}\sqrt{\frac{100}{0.020}} = 55.5n$ **per s,** $E_n = \frac{(.005\times 55.5nb_n)^2\times .08}{4} = 0.00154n^2b_n^2$ **J**

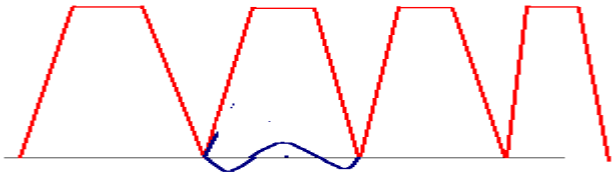
then $\mathbf{E}_1 = 2.01mJ, E_3 = 0.224mJ, E_5 = 0.081mJ, E_7 = 0.040mJ, E_9 = 0.024mJ, E_{11} = 0.015mJ$

We can also treat the traveling wave, where we derived a formula for the power passing by a given point

$P = \frac{1}{2}\mu\omega^2 A\ v$ **and we have as above**

$A_n = 0.005(\sin\frac{n\pi}{L}x)b_n$ **and** $P_n = \frac{1}{2}(0.005\sin(\frac{n\pi}{L})b_n)^2 v$ **etc. etc.**

In response to a request, what about doing a Fourier analysis on a wave form like this,with the integration running over one bump?



When I look at this suggestion, I immediately see trouble, since the integration must run over 2 pi, or the formula for the coefficients doesn't work. Indeed you can Fourier analyze the single bump, but it will not contain the fundamental mode that contains all the power! The wave equation was derived to work for a Fourier analysis from – pi to pi. When we cleverly truncate the range by a symmetry, we have to check the physical meaning to see that we don't throw the baby out with the bath. For futher explanation, see me.