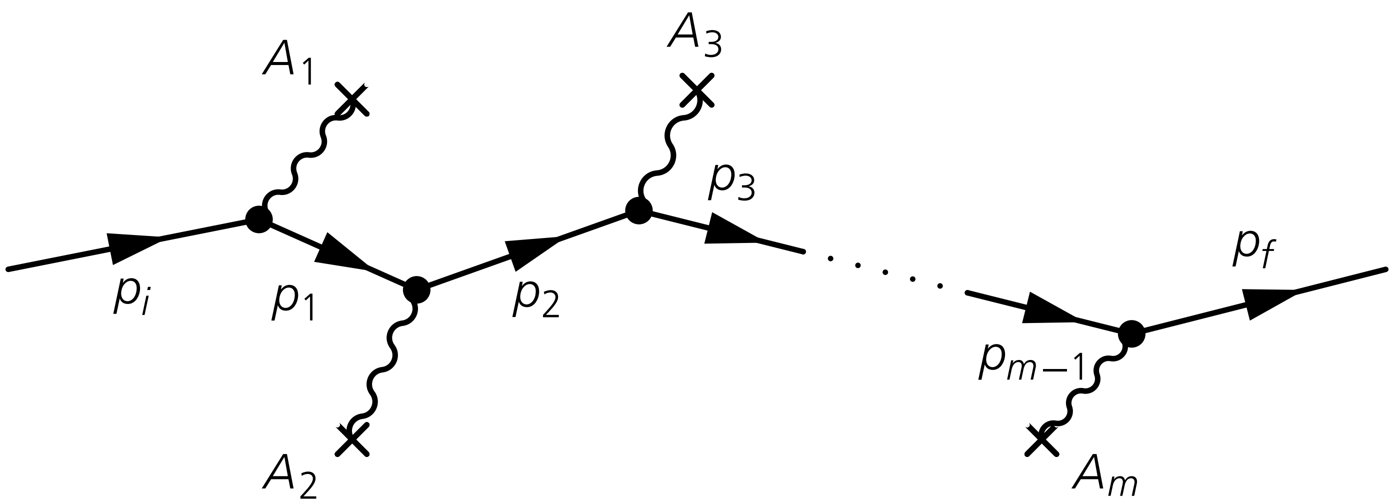
BDMPS Theory of Jet Energy Loss

Yue Shi Lai

Overview

- Radiative QCD energy loss
- Static scattering centers
- Asymptotic theory
- Closely related to Landau-Pomeranchuk-Migdal effect of multiple scatterings on bremsstrahlung

(Electromagnetic) Bremsstrahlung



- Current from four-momenta change

$$j^\mu(k) = \sum_i j_i^\mu(k) = \mathrm{ie} \sum_i \mathrm{e}^{\mathrm{i}kx_i} \left(\frac{p_i^\mu}{kp_i} - \frac{p_{i-1}^\mu}{kp_{i-1}} \right)$$

- Bremsstrahlung spectrum

$$\frac{\mathrm{d}^3 E}{\mathrm{d}^3 \boldsymbol{k}} = \omega \frac{\mathrm{d}^3 n_\gamma}{\mathrm{d}^3 \boldsymbol{k}} = \frac{1}{2(2\pi)^3} \sum_{\boldsymbol{\varepsilon}} |\boldsymbol{\varepsilon} \cdot \boldsymbol{j}(k)|^2$$

Gyulassy-Wang model

- Screened potential (QED):

$$V_i(\boldsymbol{x}) = \frac{g}{4\pi} \frac{\mathrm{e}^{-\mu|\boldsymbol{x}-\boldsymbol{x}_i|}}{\boldsymbol{q}^2+\mu^2}$$

$$A_i^\mu(\boldsymbol{q}) = \delta_0^\mu V_i(\boldsymbol{q}) = \delta_0^\mu \frac{g\mathrm{e}^{-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{x}_i}}{\boldsymbol{q}^2+\mu^2}$$

- Screened potential (QCD):

$$A_{i,A'A}^{\mu a}(\boldsymbol{q}) = T_{A'A}^a A_i^\mu(\boldsymbol{q}) = T_{A'A}^a \delta_0^\mu \frac{g\mathrm{e}^{-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{x}_i}}{\boldsymbol{q}^2+\mu^2}$$

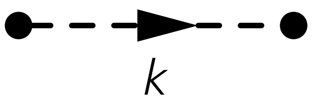
- Large mean free path limit: $\lambda \gg \mu^{-1}$
- High energy limit: $E \gg \mu$

Scalar and non-abelian vector fields (I)

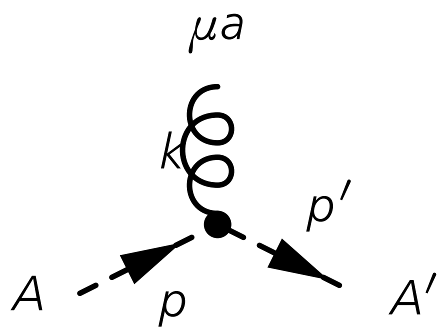
- $SU(3)$ representation: $T^a T^b = \frac{1}{2N_c}\delta^{ab} + \frac{1}{2}(d^{abc} + \mathrm{i}f^{abc})T^c$
- Spin effects in interaction with external fields are neglected, bosonic vertices are used
- Yang-Mills Lagrangian, scalar part

$$L = [(\partial^\mu - gA^{\mu a}T^a)\phi]^\dagger\,[(\partial_\mu - gA_\mu{}^a T^a)\phi] - m_\phi^2\phi^\dagger\phi$$

- $\dfrac{\mathrm{i}}{k^2 - m^2 + \mathrm{i}\eta}$

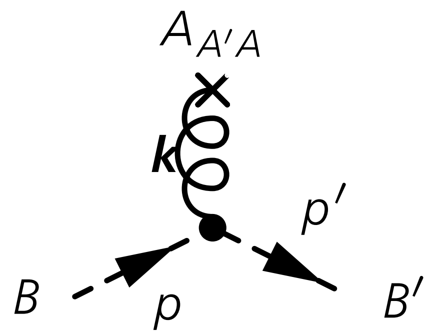


- $gT^a_{A'A}(p_\mu + p'_\mu)(2\pi)^4\delta(p - p' - k)$



Scalar and non-abelian vector fields (II)

- $\mathrm{i}g g_{\mu\nu} T^a_{B'B}(p^\mu + p'^\mu) A^{\nu a}_{A'A}(\boldsymbol{k})(2\pi)^4 \delta(p - p' - k)$

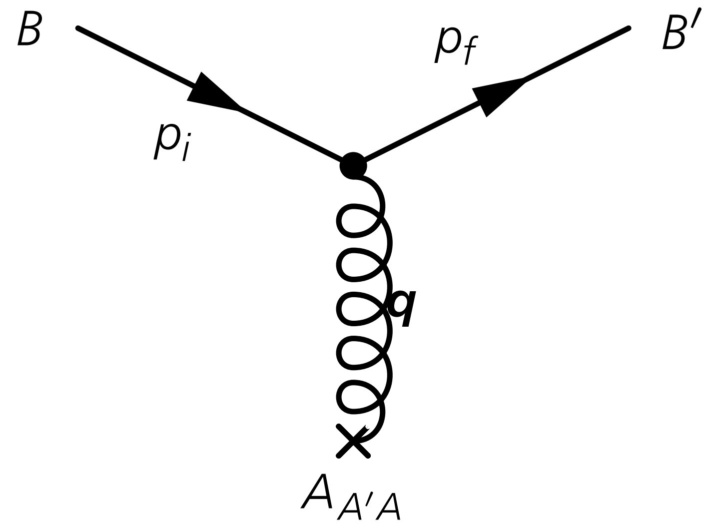


- Remember: $A^{\nu a}_{i,A'A}(\boldsymbol{k}) = T^a_{A'A} \delta^\nu_0 \dfrac{g\,\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}_i}}{\boldsymbol{k}^2 + \mu^2}$

- Total vertex factor: $\mathrm{i}g T^a_{A'A} T^a_{B'B}(p^0 + p'^0) \dfrac{g\,\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}_i}}{\boldsymbol{k}^2 + \mu^2}(2\pi)^4 \delta(p - p' - k) \equiv T^a_{B'B} M^a_{A'A}$

Gluon emission from one interaction (I)

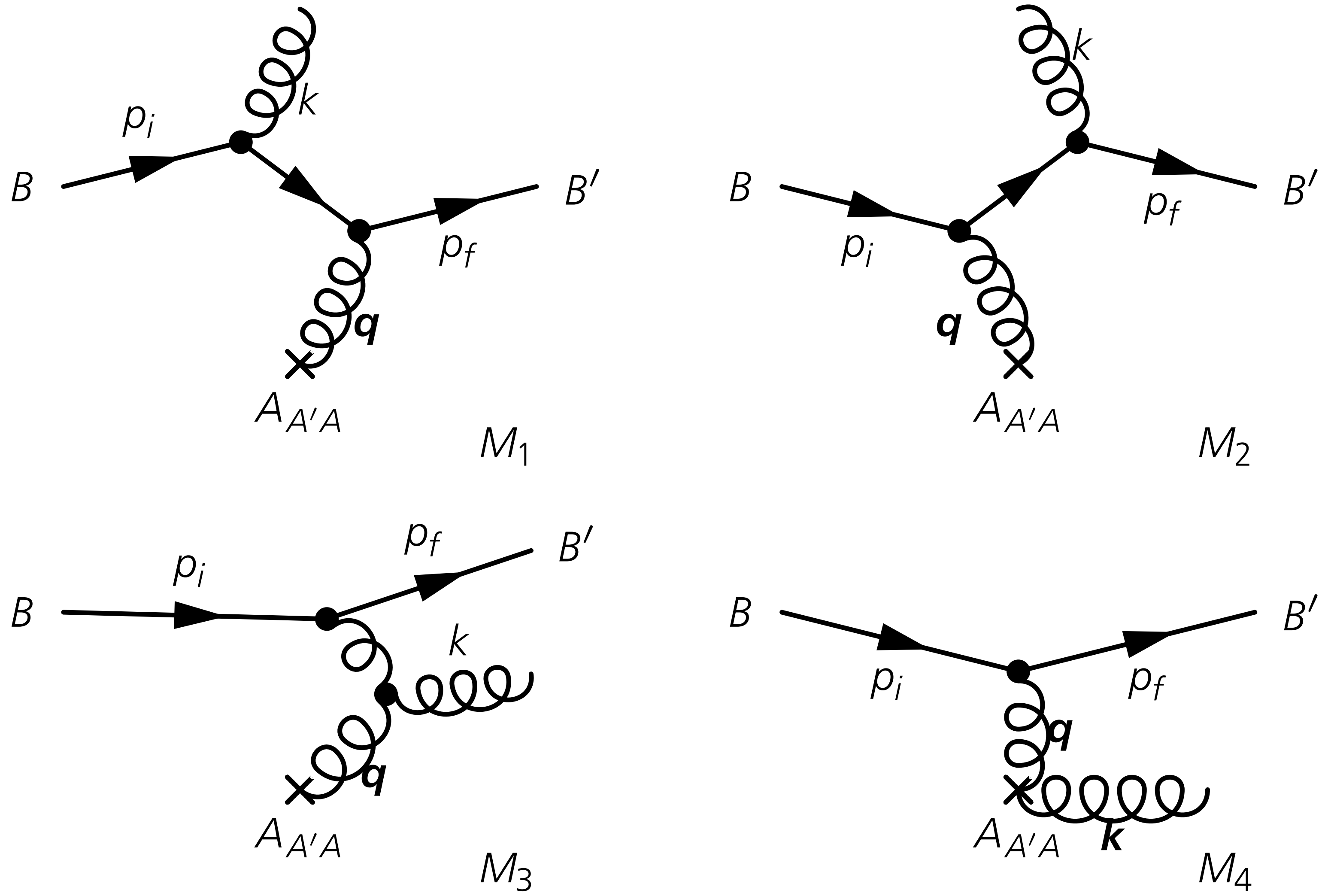
- Start with one potential interaction



- Add a gluon emission on each of the 3 lines, plus the static source
- Consider only the high energy behaviour

Gluon emission from one interaction (II)

- Resulting gluon emission graphs:



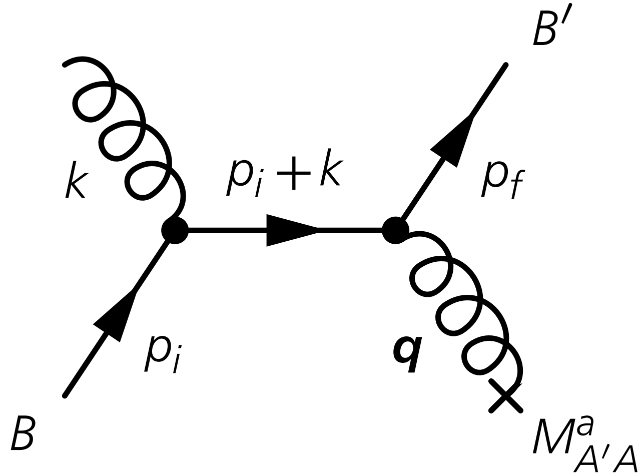
Photon emission

- $\{\gamma^{\mu},\gamma^{\nu}\} = 2g^{\mu\nu} \quad \Rightarrow \quad \not{p}\not{\varepsilon}^{*} = 2\varepsilon^{*}\cdot p - \not{\varepsilon}^{*}\not{p}$
- $k \ll p \quad \Rightarrow \quad \dfrac{\not{p}-\not{k}+m}{(p-k)^2-m^2}\not{\varepsilon}^{*} \approx \dfrac{\not{p}+m}{2pk}\not{\varepsilon}^{*} = \dfrac{\varepsilon^{*}\cdot p}{k\cdot p}$
- Emission of a photon in QED corresponds to:

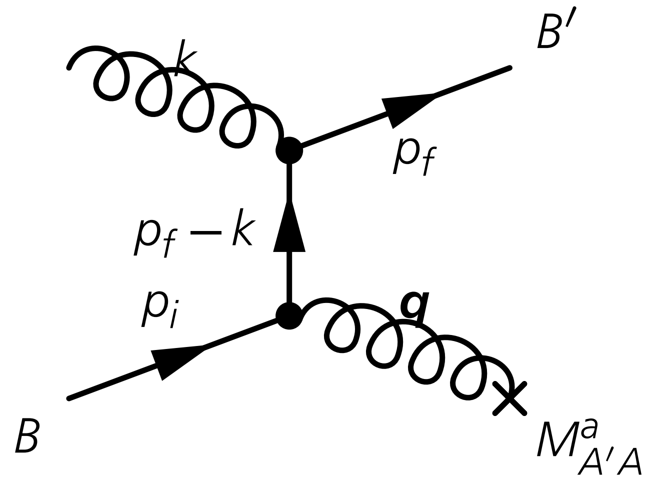
$$u(p) \quad \rightarrow \quad \frac{\mathrm{i}(\not{p}-\not{k}+m)}{(p-k)^2-m^2}(-\mathrm{i}g\underbrace{\gamma^{\nu})\varepsilon^{*}_{\nu}(k)}_{\not{\varepsilon}^{*}}u(p) = g\,\frac{\varepsilon^{*}\cdot p}{k\cdot p}u(p)$$

Gluon emission from one interaction (III)

- $$iM_1 = ig\frac{\varepsilon^* \cdot p_i}{k \cdot p_i}(T^b T^a)_{B'B} M^a_{A'A}$$



- $$iM_2 = -ig\frac{\varepsilon^* \cdot p_f}{k \cdot p_f}(T^a T^b)_{B'B} M^a_{A'A}$$



Gluon emission from one interaction (IV)

- In light cone gauge:

$$\varepsilon = (\varepsilon^0, -\varepsilon^0, \boldsymbol{\varepsilon}_\perp)$$

$$p = (p^0, p_{||}, \boldsymbol{p}_\perp)$$

- Chose $\varepsilon \cdot k = 0$:

$$\varepsilon^0(k^0 + k_{||}) - \boldsymbol{\varepsilon}_\perp \cdot \boldsymbol{k}_\perp = 0$$

$$\varepsilon^0 = \frac{\boldsymbol{\varepsilon}_\perp \cdot \boldsymbol{k}_\perp}{k^0 + k_{||}} \approx \frac{\boldsymbol{\varepsilon}_\perp \cdot \boldsymbol{k}_\perp}{2k^0}$$

- Taking $(p^0)^2 \gg p_\perp^2$:

$$p_{||} \approx p^0 - \frac{p_\perp^2}{2p^0}$$

$$\varepsilon \cdot p = \varepsilon^0(p^0 + p_{||}) - \boldsymbol{\varepsilon}_\perp \cdot \boldsymbol{p}_\perp \approx p^0 \left(2\varepsilon^0 - \frac{\boldsymbol{\varepsilon}_\perp \cdot \boldsymbol{p}_\perp}{p^0}\right) = p^0 \boldsymbol{\varepsilon}_\perp \left(\frac{\boldsymbol{k}_\perp}{k^0} - \frac{\boldsymbol{p}_\perp}{p^0}\right)$$

Gluon emission from one interaction (V)

- Same for $k \cdot p$:

$$k \cdot p = k^0 p^0 \left[1 - \left(1 - \frac{k_\perp^2}{2(k^0)^2}\right)\left(1 - \frac{p_\perp^2}{2(p^0)^2}\right) - \frac{\boldsymbol{k}_\perp \cdot \boldsymbol{p}_\perp}{k^0 p^0} \right]$$

$$= \frac{k^0 p^0}{2} \left(\frac{k_\perp^2}{(k^0)^2} + \frac{p_\perp^2}{(p^0)^2} - \frac{2\boldsymbol{k}_\perp \cdot \boldsymbol{p}_\perp}{k^0 p^0} - \frac{k_\perp^2 p_\perp^2}{(k^0)^2 (p^0)^2} \right)$$

$$\approx \frac{k^0 p^0}{2} \left(\frac{\boldsymbol{k}_\perp}{k^0} - \frac{\boldsymbol{p}_\perp}{p^0} \right)^2$$

- Compare last two results

$$\frac{\varepsilon \cdot p}{k \cdot p} = \frac{2\boldsymbol{\varepsilon}_\perp}{k^0} \frac{\boldsymbol{k}_\perp / k^0 - \boldsymbol{p}_\perp / p^0}{\left(\boldsymbol{k}_\perp / k^0 - \boldsymbol{p}_\perp / p^0\right)^2} \approx \frac{2\boldsymbol{\varepsilon}_\perp \cdot \boldsymbol{k}_\perp}{k_\perp^2}$$

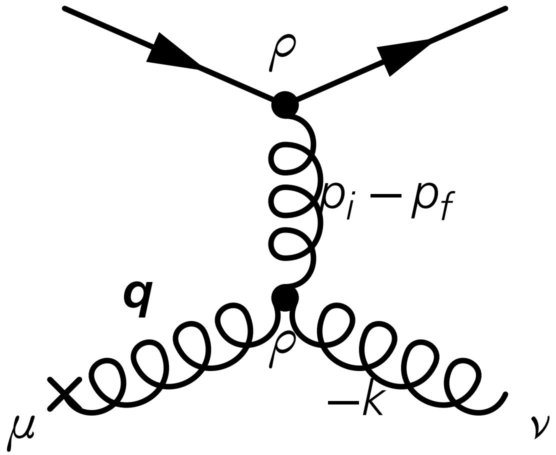
- Combine M_1, M_2:

$$\mathrm{i}(M_1 + M_2) = -2\mathrm{i}g \frac{\boldsymbol{\varepsilon}_\perp \cdot \boldsymbol{k}_\perp}{k_\perp^2} [T^b, T^a]_{B'B} M^a_{A'A}$$

Gluon emission from one interaction (VI)

- For M_3, approximate $q \approx -k$, $p_f - p_i \ll k$, 3-gluon vertex:

$$\Lambda_{abc} \propto f_{abc}\left[g_{\mu\nu}\underbrace{(q+k)_\sigma}_{(p_f-p_i+2k)_\sigma}\varepsilon^\sigma + \underbrace{(p_f-p_i-k)_\mu}_{(q-2k)_\mu}\varepsilon_\nu + \underbrace{(p_i-p_f-k)_\nu}_{(k-2q)_\nu}\varepsilon^\mu\right]$$



- Whole diagram:

$$iM_3 \approx -ig\frac{2}{(p_f-p_i)^2}\left[g_{\mu\nu}\varepsilon\cdot(p_f-p_i) - k_\mu\varepsilon_\nu + k_\nu\varepsilon_\mu\right]M^{a,\mu\nu}_{A'A}[T^b,T^a]_{B'B}$$

$$= 2ig\boldsymbol{\varepsilon}_\perp\cdot\frac{\boldsymbol{k}_\perp-\boldsymbol{q}_\perp}{(\boldsymbol{k}_\perp-\boldsymbol{q}_\perp)^2}[T^b,T^a]_{B'B}M^a_{A'A}$$

Gluon emission from one interaction (VII)

- M_4 is suppressed by $k_\perp/k^0$ and neglected
- $M_1 + M_2$, M_3 share the common factor $2g\boldsymbol{\varepsilon}_\perp[T^b, T^a]_{B'B}M^a_{A'A}$
- Define

$$\boldsymbol{J}(k,q) = \frac{\boldsymbol{k}_\perp}{k_\perp^2} - \frac{\boldsymbol{k}_\perp - \boldsymbol{q}_\perp}{(\boldsymbol{k}_\perp - \boldsymbol{q}_\perp)^2}$$

$$\boldsymbol{J}_{\text{eff}}(k,q) = \boldsymbol{J}(k,q)[T^b, T^a]_{B'B}$$

$$\mathrm{d}\sigma_{\text{rad}} \propto N_c \frac{\alpha_s}{\pi^2} \boldsymbol{J}^2 \underbrace{C_F |M^a|^2 \mathrm{d}q_\perp^2}_{\mathrm{d}\sigma_{\text{el}}} \mathrm{d}^2\boldsymbol{k}_\perp \frac{\mathrm{d}k_\parallel}{k^0} \qquad |M^a|^2 \delta^{aa'} \equiv \frac{\mathrm{tr} M^a M^{a'}}{N_c}$$

- The gluon energy spectrum

$$\omega \frac{\mathrm{d}^3 I}{\mathrm{d}\omega \mathrm{d}^2\boldsymbol{k}_\perp} = N_c \frac{\alpha_s}{\pi^2} \left\langle \boldsymbol{J}^2(k,q) \right\rangle = \frac{\alpha_s}{\pi^2} \frac{\left\langle \boldsymbol{J}_{\text{eff}}^2(k,q) \right\rangle}{C_F N_c} \qquad \langle \cdot \rangle \equiv \int \mathrm{d}\boldsymbol{q}_\perp V(q_\perp^2)(\cdot)$$

Gluon emission from multiple interactions (I)

- Phase factor shorthand: $\phi_i(k) = \frac{t_i k_\perp}{2\omega}$
- By calculating diagrams with higher number of interactions, one realizes:

$$\omega \frac{dI^{(N)}}{d\omega} = \frac{\alpha_s}{\pi^2} \int d^2\boldsymbol{k}_\perp \left\langle \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{\boldsymbol{J}_{\text{eff},i} \cdot \boldsymbol{J}^\dagger_{\text{eff},j}}{N_c C_F^N} e^{i(\phi_i - \phi_j)} \right\rangle$$

$$= \frac{\alpha_s}{\pi^2} \int d^2\boldsymbol{k}_\perp \left\langle 2\mathrm{Re} \sum_{i=1}^{N} \sum_{j=i+1}^{N} \frac{\boldsymbol{J}_{\text{eff},i} \cdot \boldsymbol{J}^\dagger_{\text{eff},j}}{N_c C_F^{j-i+1}} e^{i(\phi_i - \phi_j)} + \sum_{i=1}^{N} |J_{\text{eff},i}|^2 \right\rangle$$

$$= \frac{\alpha_s}{\pi^2} \int d^2\boldsymbol{k}_\perp \left\langle 2\mathrm{Re} \sum_{i=1}^{N} \sum_{j=i+1}^{N} \frac{\boldsymbol{J}_{\text{eff},i} \cdot \boldsymbol{J}^\dagger_{\text{eff},j}}{N_c C_F^{j-i+1}} \left[e^{i(\phi_i - \phi_j)} - 1 \right] + \underbrace{\left| \sum_{i=1}^{N} J_{\text{eff},i} \right|^2}_{\text{factorization term}} \right\rangle$$

- It can be also shown that the factorization term has no medium dependency, and is going to be neglected

Solution for Infinite Plasma (I)

- Introduce "by finger" the fermion survival probability:

$$\langle\cdot\rangle \to \int \prod_{l=1}^{N-1} \frac{\mathrm{d}(z_l - z_{l-1})}{\lambda} \exp\left(-\frac{z_l - z_{l-1}}{\lambda}\right) \mathrm{d}\boldsymbol{q}_{\perp,i} N V(q_{\perp,i}^2)(\cdot)$$

(normalization $N = \mu^2/\pi$ for Gyulassy-Wang)

- Rescale $\boldsymbol{U} = \frac{\boldsymbol{k}_\perp}{\mu}, \boldsymbol{Q} = \frac{\boldsymbol{q}_\perp}{\mu}$, define $\varkappa = \frac{\lambda\mu^2}{2}\frac{1}{\omega}$
- $\boldsymbol{U}_i = \boldsymbol{U}_{i-1} - \boldsymbol{Q}_i, \quad \boldsymbol{J}_i = \frac{\boldsymbol{U}_i}{U_i^2} - \frac{\boldsymbol{U}_{i-1}}{U_{i-1}^2}$
- For infinite plasma, one considers $N \to \infty$, differential spectrum per unit length:

$$\omega\frac{\mathrm{d}I}{\mathrm{d}\omega\mathrm{d}z} = \frac{3}{2}N_c\frac{\alpha_s}{\pi^2}\int \mathrm{d}^2\boldsymbol{U} \left\langle 2\mathrm{Re}\sum_{j=2}^{\infty} \boldsymbol{J}_1\cdot\boldsymbol{J}_j \left[\exp\left(\mathrm{i}\varkappa\sum_{l=1}^{j+1} U_l^2 \frac{z_l - z_{l-1}}{\lambda}\right) - 1\right]\right\rangle$$

Solution for Infinite Plasma (II)

- Evaluating $\langle\cdot\rangle$ gives:

$$\omega\frac{\mathrm{d}I}{\mathrm{d}\omega\mathrm{d}z}=\frac{3}{2}N_c\frac{\alpha_s}{\pi\lambda}\int\frac{\mathrm{d}^2\boldsymbol{U}}{\pi}2\mathrm{Re}\sum_{j=0}^{\infty}\int\prod_{l=1}^{j+2}d^2\boldsymbol{Q}_l V(Q_l^2)\boldsymbol{J}_1\boldsymbol{J}_{n+2}\left[\prod_{m=1}^{j+1}\psi(U_m^2)-1\right]$$

with $\psi(U^2)=(1-i\varkappa U^2)^{-1}$

- It is now a purely mathematical problem to solve this integral (which in general does not solve algebraically)
- Integrating over $\boldsymbol{Q}_i$:

$$\boldsymbol{J}_1\cdot\boldsymbol{J}_{i+2}=\left(\frac{\boldsymbol{U}_1}{U_1^2}-\frac{\boldsymbol{U}_1+\boldsymbol{Q}_1}{(\boldsymbol{U}_1+\boldsymbol{Q}_1)^2}\right)\cdot\left(\frac{\boldsymbol{U}_{i+1}-\boldsymbol{Q}_{i+2}}{(\boldsymbol{U}_{i+1}+\boldsymbol{Q}_{i+2})^2}-\frac{\boldsymbol{U}_{i+1}}{U_{i+1}^2}\right)$$

Solution for Infinite Plasma (II)

- Note $\int \mathrm{d}^2\boldsymbol{Q} V(Q^2) \frac{\boldsymbol{U}-\boldsymbol{Q}}{(\boldsymbol{U}-\boldsymbol{Q})^2} = \pi \frac{\boldsymbol{U}}{U^2} \int_0^{U^2} \mathrm{d}Q^2 V(Q^2)$
  (trivial, spherically symmetric distributions do not have multipole moments)
- Define two current averages:

$$\boldsymbol{f}_0(\boldsymbol{U}_1) \equiv \int \mathrm{d}^2\boldsymbol{Q}_1 V(Q_1^2) \boldsymbol{J}_1 = \pi \frac{\boldsymbol{U}_1}{U_1^2} \int_{U_1^2}^{\infty} \mathrm{d}Q^2 V(Q^2)$$

$$\boldsymbol{f}_0(\boldsymbol{U}_{i+1}) \equiv -\int \mathrm{d}^2\boldsymbol{Q}_{i+2} V(Q_{i+2}^2) \boldsymbol{J}_1 = \pi \frac{\boldsymbol{U}_{i+1}}{U_{i+1}^2} \int_{U_{i+1}^2}^{\infty} \mathrm{d}Q^2 V(Q^2)$$

Solution for Infinite Plasma (III)

- Rewritten spectrum:

$$\omega\frac{\mathrm{d}I}{\mathrm{d}\omega\mathrm{d}z}=\frac{2\alpha}{\pi\lambda}\mathrm{Re}\int\frac{\mathrm{d}\boldsymbol{U}_1}{\pi}\boldsymbol{f}_0(\boldsymbol{U}_1)\cdot\boldsymbol{f}(\boldsymbol{U}_1)\Big|_{\varkappa}^{\varkappa=0}$$

$$\boldsymbol{f}(\boldsymbol{U}_1)=\psi(U_1^2)\boldsymbol{f}_0(\boldsymbol{U}_1)+\psi(U_1^2)\sum_{i=1}^{\infty}\prod_{j=2}^{i+1}\left[\int\mathrm{d}^2Q_jV(Q_j^2)\psi(U_j^2)\right]\boldsymbol{f}_0(\boldsymbol{U}_{i+1})$$

- Resulting integral equation for $\boldsymbol{f}$:

$$(1-\mathrm{i}\varkappa U^2)\boldsymbol{f}(\boldsymbol{U})=\boldsymbol{f}_0(\boldsymbol{U})+\int\mathrm{d}^2\boldsymbol{Q}V(Q^2)\boldsymbol{f}(\boldsymbol{U}-\boldsymbol{Q})$$

- Solution for infinite plasma in the leading log approximation (LLA), $\varkappa\ll 1$

$$\omega\frac{\mathrm{d}I}{\mathrm{d}\omega\mathrm{d}z}=\frac{3\alpha_s}{2\pi}C_R\sqrt{\frac{\mu}{\lambda_g\omega}\ln\left(\frac{\omega}{\mu^2\lambda_g}\right)}$$

Finite Plasma, Energy Loss

- Solution for finite plasma, $\varkappa \ll 1$:

$$\omega \frac{\mathrm{d}I}{\mathrm{d}\omega \mathrm{d}z} = \frac{6\alpha_s}{\pi L} C_R \ln \left| \frac{\sin \omega_0 \tau_0}{\omega_0 \tau_0} \right| \qquad \tau_0 = \frac{N_c}{2C_R} \frac{L}{\lambda_R} \qquad \omega_0 = \sqrt{\mathrm{i} \frac{2C_R}{N_c} \frac{\mu^2 \lambda_R}{2\omega} \cdot \mathrm{const.}}$$

- Relation between radiation spectrum and energy loss

$$-\frac{\mathrm{d}E}{\mathrm{d}z} = \int \mathrm{d}\omega \, \omega \frac{\mathrm{d}I}{\mathrm{d}\omega \mathrm{d}z}$$

- Energy loss

$$-\frac{\mathrm{d}E}{\mathrm{d}z} = \frac{\alpha_s C_R}{8} \frac{\mu}{\lambda_g} L^2 \ln \frac{L}{\lambda_g}$$

Numerical Estimates, compare with GLV

- $T = 250$ MeV, $\mu^2/\lambda = 1$ GeV/fm^2, $\alpha_s = 1/3$: $-\Delta E \approx 30\,\text{GeV}\left(\frac{L}{10\,\text{fm}}\right)^2$ (too large)
- GLV diagrams are evaluated like BDMPS
- GLV generates more diagrams, equivalence at the level of two scatterings
- BDMPS is a highly simplified theory with $1/\mu \ll \lambda \ll L \ll E/\mu^2$ approximation
- Only angularly integrated energy distribution $\mathrm{d}I/\mathrm{d}\omega$ is available
- B. G. Zaharov (2000): "LLA fails when the gluon formation length becomes of the order of L"
- Another formalism that might be of interest: B. G Zaharov, hep-ph/9607440