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Multistage pulse tubes

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Abstract

In this paper we address the question of when and how multistaging in pulse-tube refrigerators improves the performance. A two-stage pulse-tube refrigerator is treated as an example. In order to avoid complicated mathematical or numerical calculations we assume that the only irreversible process in the regenerator is heat conduction and that the average enthalpy flow in the regenerator is zero. We derive analytical expressions for the position of the first stage connection to the regenerator in the case of maximum cooling power and in the case of the minimum temperature.

1. Introduction

Many types of cryocoolers for the lower temperature region are carried out as coolers in series. One cooler is used to precool the next stage. However, a precooling stage reduces the flow to the final stage, hence has the tendency to reduce the cooling power. In this paper we address the question of when and how multistaging in pulse tube coolers improves the coefficient of performance and under which conditions this improvement can be optimized. In order to avoid complicated mathematical or numerical calculations we assume that the only irreversible process in the regenerator is heat conduction. So the flow resistance of the regenerator is zero and the thermal contact between the fluid and the matrix is perfect. If in addition the heat capacity of the regenerator matrix is very large (so the gas temperature is constant in time but varies with position) and the working fluid is an ideal gas then, in the steady state, the average enthalpy flow in the regenerator is zero.

First we discuss the entropy production due to heat conduction. Next we will discuss the single-stage pulse-tube refrigerator (PTR) and then the two-stage PTR.

2. Heat conduction

As the only irreversible process taken into account is thermal conduction, the entropy production per unit length is given by [1]

$$\frac{\mathrm{d}\dot{S}_{\mathrm{i}}}{\mathrm{d}l} = -\frac{\dot{Q}}{T^2}\frac{\mathrm{d}T}{\mathrm{d}l}. \quad (1)$$

The total entropy production in the regenerator is

$$\dot{S}_{\mathrm{ir}} = \int_{T_{\mathrm{L}}}^{T_{\mathrm{H}}} \frac{\dot{Q}}{T^2}\,\mathrm{d}T. \quad (2)$$

Here T_{H} is the temperature at the hot end ($l = 0$) and T_{L} the temperature at the cold end ($l = L$). The heat flow in a regenerator with thermal conductivity κ and area A is given by

$$\dot{Q} = -\kappa A\frac{\mathrm{d}T}{\mathrm{d}l}. \quad (3)$$

In the presence of cooling stages along the regenerator $\dot{Q}$ need not be constant. At each connection of a stage $\dot{Q}$ changes. With a very large number of stages $\dot{Q}$ is a continuous function of position. We can ask which $\dot{Q}(l)$ dependence would give the minimum entropy production with the restriction that the integral

$$L = -\int_{T_{\mathrm{L}}}^{T_{\mathrm{H}}} \frac{\mathrm{d}l}{\mathrm{d}T}\mathrm{d}T = \int_{T_{\mathrm{L}}}^{T_{\mathrm{H}}} \frac{\kappa A}{\dot{Q}}\,\mathrm{d}T \quad (4)$$

has a fixed value. This is a typical variational problem. We introduce λ, which has the same dimension as the thermal conductivity κ, as a Lagrange multiplier and look for the $\dot{Q} - T$ dependence which gives a minimum of the integral in Eq. (2) for a fixed value of the integral in Eq. (4). Hence we require that for all variations $\delta\dot{Q}(T)$ of the function $\dot{Q}(T)$

$$\delta\int_{T_{\mathrm{L}}}^{T_{\mathrm{H}}} \left(\frac{\dot{Q}}{T^2} + \lambda\frac{\kappa A}{\dot{Q}}\right)\mathrm{d}T = 0. \quad (5)$$

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Nomenclature

A	area (m^2)
k	dimensionless heat conductivity
l	axial length parameter (m)
L	length of the regenerator (m)
O	orifice
P	power (W)
q	dimensionless heat flow
Q	heat (J)
S	entropy (J/K)
t	dimensionless temperature
T	temperature (K)
x	dimensionless distance
κ	coefficient of thermal conductivity (W/K m)
λ	Lagrange parameter
ψ	defined in Eq. (50)

Subscripts

0	value at optimum
1	first stage
2	second stage
B	at base temperature
H	hot (room-temperature) end
i	irreversible
L	single-stage low-temperature end
lin	linear T-profile
m	at minimum of ψ
O	orifice
r	regenerator

Eq. (5) gives that

$$\frac{1}{T^2} - \lambda \frac{\kappa A}{\dot{Q}_0^2} = 0 \tag{6}$$

or

$$\dot{Q}_0 = \sqrt{\lambda}\sqrt{\kappa A}T, \tag{7}$$

where $\dot{Q}_0(T)$ is the optimum $\dot{Q} - T$ dependance. In general κA is a function of T. If we assume κA to be constant, then substitution in Eq. (4) gives

$$\lambda = \frac{\kappa A}{L^2} \ln^2 \frac{T_H}{T_L} \tag{8}$$

so

$$\dot{Q}_0 = \kappa A \frac{T}{\mathrm{L}} \ln \frac{T_H}{T_L}. \tag{9}$$

This is the heat flow pattern that generates the minimum amount of entropy while still satisfying Eq. (4). The temperature profile can now be found by substituting Eq. (9) in Eq. (3) which gives a differential equation with solution

$$T(l) = T_H \left(\frac{T_L}{T_H}\right)^{l/L}. \tag{10}$$

In dimensionless form, introducing $t = T/T_H$ and $x = l/L$, this reads

$$t(x) = t_L^x. \tag{11}$$

The total entropy production is

$$\dot{S}_{ir} = \frac{\kappa A}{\mathrm{L}} \ln^2 \frac{T_H}{T_L}. \tag{12}$$

This is the minimum entropy production of a homogeneous tube spanning a temperature range from T_H to T_L. This may be compared with the entropy production of a linear T-profile

$$\dot{S}_{lin} = \frac{\kappa A}{\mathrm{L}} T_H \left(\frac{1}{T_L} - \frac{1}{T_H}\right). \tag{13}$$

3. Single-stage PTR

We start from the expression for the power P of the compressor for the single-stage PTR in the steady state [2]

$$P = \left(\frac{T_H}{T_L} - 1\right)\dot{Q}_L + T_H\left(\dot{S}_r + \dot{S}_O\right). \tag{14}$$

Here $\dot{Q}_L$ the externally applied cooling power at the temperature T_L. The heat flow, which flows inside the regenerator, affects the entropy production terms but it does not show up explicitly in Eq. (14). The total entropy production is the sum of the entropy production due to heat conduction in the regenerator

$$\dot{S}_r = \dot{Q}\left(\frac{1}{T_L} - \frac{1}{T_H}\right) \tag{15}$$

and the entropy production of the orifice

$$\dot{S}_O = \frac{\dot{Q} + \dot{Q}_L}{T_H}. \tag{16}$$

Substitution of Eqs. (15) and (16) in Eq. (14) gives for the coefficient of performance of the single-stage pulse tube

$$\mathrm{COP}_1 = \frac{\dot{Q}_L}{P} = \frac{T_L}{T_H} - \frac{\dot{Q}}{P}. \tag{17}$$

If κA is constant, then the heat flow is given by

$$\dot{Q} = \frac{\kappa A(T_H - T_L)}{\mathrm{L}}. \tag{18}$$

Writing

$$k = \frac{\kappa A T_H}{LP}, \tag{19}$$

which can be regarded as the dimensionless heat conductivity, and introducing the dimensionless heat flow

$$q = \frac{\dot{Q}}{P} \tag{20}$$

gives, with Eq. (18),

$$q = k(1 - t_L). \tag{21}$$

The COP can be expressed as

$$\mathrm{COP}_1 = t_L - k + kt_L. \tag{22}$$

In the ideal case $\mathrm{COP}_1 = t_L$ so the last terms express the loss in cooling power due to the heat conduction. At the base temperature $\dot{Q}_L = 0$, so $\mathrm{COP}_1 = 0$. From Eq. (22) the single-stage base temperature is given by

$$t_B = \frac{k}{1+k}. \tag{23}$$

4. Two-stage PTR

4.1. General expressions

A schematic picture of the two-stage pulse tube is represented in Fig. 1. We start from the expression analogous to Eq. (14)

$$P = \left(\frac{T_H}{T_2} - 1\right)\dot{Q}_L + T_H\left(\dot{S}_{r1} + \dot{S}_{r2} + \dot{S}_{O1} + \dot{S}_{O2}\right). \tag{24}$$

Here $\dot{Q}_L$ the net cooling power at the second stage temperature T_2. The entropy production due to heat conduction in the first part of the regenerator is

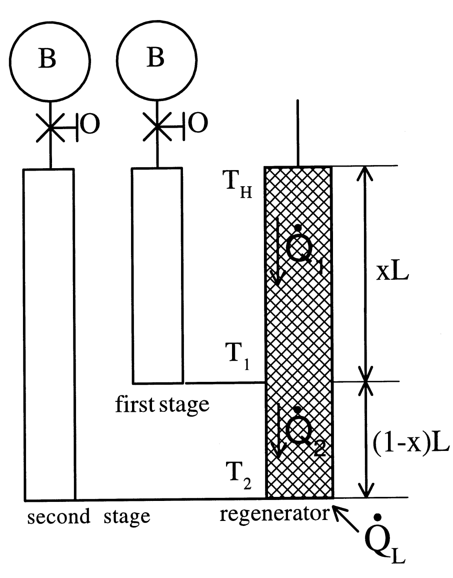

Fig. 1. Schematic diagram of a two-stage PTR. The total length of the regenerator is L. The first stage is connected at position xL.

$$\dot{S}_{r1} = \dot{Q}_1\left(\frac{1}{T_1} - \frac{1}{T_H}\right) \tag{25}$$

and in the second part

$$\dot{S}_{r2} = \dot{Q}_2\left(\frac{1}{T_2} - \frac{1}{T_1}\right). \tag{26}$$

The entropy production of the first-stage orifice satisfies

$$T_H\dot{S}_{O1} = \dot{Q}_1 - \dot{Q}_2 \tag{27}$$

and of the second-stage orifice

$$T_H\dot{S}_{O2} = \dot{Q}_2 + \dot{Q}_L. \tag{28}$$

Summing these four contributions gives with Eq. (24)

$$P = \frac{T_H}{T_2}\dot{Q}_L + \dot{Q}_1\frac{T_H}{T_1} + \dot{Q}_2\left(\frac{T_H}{T_2} - \frac{T_H}{T_1}\right). \tag{29}$$

If $\dot{Q}_1 = \dot{Q}_2$ we get back the expression for the single-stage PTR (Eq. (14)). Between room temperature and the first stage at temperature T_1 flows a heat $\dot{Q}_1$ given by

$$\dot{Q}_1 = \frac{\kappa_1 A_1 (T_H - T_1)}{L_1}. \tag{30}$$

In the second stage flows a heat

$$\dot{Q}_2 = \frac{\kappa_2 A_2 (T_1 - T_2)}{L_2}. \tag{31}$$

For physically significant results we require

$$T_H \geqslant T_1 \geqslant T_2 \geqslant 0 \tag{32}$$

and

$$\dot{Q}_1 \geqslant \dot{Q}_2. \tag{33}$$

With the dimensionless heat conductivities

$$k_i = \frac{\kappa_i A_i T_H}{L_i P} \quad (i = 1, 2) \tag{34}$$

the COP

$$\mathrm{COP}_2 = \frac{\dot{Q}_L}{P} \tag{35}$$

the dimensionless heat flows

$$q_i = \frac{\dot{Q}_i}{P} \tag{36}$$

and the dimensionless temperatures

$$t_i = \frac{T_i}{T_H}. \tag{37}$$

Eqs. (30) and (31) are written as

$$q_1 = k_1(1 - t_1), \tag{38}$$

$$q_2 = k_2(t_1 - t_2). \tag{39}$$

With Eq. (29) we find the expression for the COP in terms of the reduced heat conductivities (k_1 and k_2) and the reduced temperatures

$$COP_2 = t_2 - k_2 t_1 - \frac{k_1 t_2 + k_2 t_2^2}{t_1} + (k_1 + 2k_2)t_2. \quad (40)$$

First we discuss the optimum COP_2 for variation of t_1. This optimization can be arranged by adjustment of the orifice of the first stage, assuming a fixed configuration of the double-inlet PTR, i.e., fixed k_1 and k_2. Hence in the optimum situation

$$t_{10} = \sqrt{\frac{k_1 t_2 + k_2 t_2^2}{k_2}}. \quad (41)$$

Inserting in Eq. (40) gives the optimum COP_2

$$COP_{20} = t_2 - 2k_2 t_{10} + (k_1 + 2k_2)t_2. \quad (42)$$

Now we will determine the best point to connect the first stage to the regenerator. For this we put

$$k_1 = \frac{k}{x} \quad \text{and} \quad k_2 = \frac{k}{1-x}, \quad (43)$$

which can be regarded as expressions defining two new variables k and x. In a simple case Eq. (43) corresponds with a regenerator with a total length $L = L_1 + L_2$ with the length of the first part of the regenerator $L_1 = xL$, and of the second part $L_2 = (1-x)L$ (Fig. 1) and $\kappa_1 A_1 = \kappa_2 A_2 = \kappa A$ so that

$$k = \frac{\kappa A T_H}{LP}. \quad (44)$$

With Eq. (43), Eq. (41) obtains the form

$$t_{10}(x, t_2) = \sqrt{\frac{1-x}{x} t_2 + t_2^2}. \quad (45)$$

The conditions in Eq. (32) are satisfied if

$$\frac{t_2}{1 + t_2 - t_2^2} \leqslant x. \quad (46)$$

The reduced heat flows (38) and (39), in the optimum situation, read

$$q_{10} = \frac{k}{x}(1 - t_{10}) \quad (47)$$

and

$$q_{20} = \frac{k}{1-x}(t_{10} - t_2). \quad (48)$$

The COP is derived from Eq. (42) and results in the important expression

$$COP_{20} = t_2 - k + k\psi \quad (49)$$

with

$$\psi(x, t_2) = 1 - 2\sqrt{\frac{t_2}{x(1-x)} + \left(\frac{t_2}{1-x}\right)^2} + \frac{t_2}{x} + \frac{2t_2}{1-x}. \quad (50)$$

The function ψ contains the information of the variation of the COP as function of x, the position of the first stage. In the following sections we will discuss the implications of Eq. (50) for the performance and the design of two-stage PTRs.

4.2. Maximum cooling power

Some examples of $\psi - x$ dependencies for various values of T_2 are given in Fig. 2. In the limit of $x = 1$ we find $\psi(1, t_2) = t_2$, which is the value for a single stage PTR (Eq. (22)). For fixed t_2 the function ψ has a minimum at

$$x_m = \frac{t_2}{1 - t_2}. \quad (51)$$

Substitution in Eqs. (45), (47) and (48) gives for the reduced first-stage temperature

$$t_{1m} = 1 - t_2 \quad (52)$$

and for the reduced heat flows at the minimum of ψ

$$q_{1m} = q_{2m} = k t_{1m}. \quad (53)$$

In this situation $q_{1m} = q_{2m}$, so the cooling power of the first stage is zero. Basically, the system is a single-stage PTR. Substitution of x_m in Eq. (50) shows $\psi(x_m, t_2) = t_2$ which indeed is the same value as of a single stage PTR (Eq. (22)). For $x < x_m$ we have $q_{1m} < q_{2m}$ which is not physically meaningful (condition (33)).

For maximum cooling power at some given t_2 the value of x must be chosen in such a way that the function $\psi(x, t_2)$ is a maximum. For $x_m \leqslant x \leqslant 1$ the function ψ has a maximum at

$$x_0 = \frac{4 - t_2 + \sqrt{t_2^2 + 4t_2}}{8 - 6t_2}, \quad (54)$$

which gives a larger COP than for the single-stage COP. The x_0-value given by Eq. (54) satisfies condition (46) and ranges from 0.5 (for $t_2 = 0$) to 1 (for $t_2 = 0.5$). In

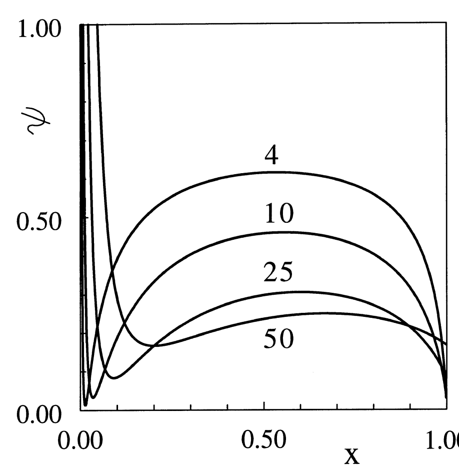

Fig. 2. The function ψ for various values of T_2 (indicated in the figure) as functions of x. The preferred position of the first stage is at the maximum of ψ. For higher T_2 the maximum shifts to higher x values, which means that the connection point is closer to the low-temperature end of the regenerator.

engineering terms this means that, for high temperatures ($T_2 > 0.5T_H$), a double stage has no positive effect, that for very low T_2-values the first stage should be connected to the middle of the regenerator ($x = 0.5$), and that for T_2 just below $0.5T_H$ the first stage should be connected near the cold end ($x = 1$). Near the maximum the $\psi - x$ curves are fairly flat which means that in a practical situation the engineer has some freedom to connect the first stage at a convenient position.

Inserting Eq. (54) in Eq. (45) gives the relation for the first-stage temperature in the position for maximum cooling power

$$t_{100} = t_{10}(x_0). \tag{55}$$

We must demand $t_{100} \geqslant t_2$, which means $t_2 \leqslant 1/2$. In the optimum situation condition (46) is always satisfied.

4.3. Minimum temperature

The base temperature t_{B2} of the system is the second-stage temperature with $\dot{Q}_L = 0$. Putting $COP_{20} = 0$ in Eq. (49) gives

$$t_{B2}(x, k) = \frac{4xk^2}{(x+k)(k+3kx+x-x^2)}. \tag{56}$$

As an example a plot of t_{B2} as function of x is presented in Fig. 3 for $k = 0.1$. Analysis of Eq. (56) shows that, at $x = k$, t_{2mi} has a maximum equal to $k/(1+k)$, which corresponds with the single-stage PTR. In this point $q_{10} = q_{20} = k/(1+k)$. The minimum value of t_{B2}, for k fixed, is the lowest two-stage base temperature and will be called t_{B20}. It is found at the position given by

$$x_{B0}(k) = \frac{1}{4}\left(1 + \sqrt{1+8k}\right). \tag{57}$$

As $x_{B0} \leqslant 1$ it follows that $k \leqslant 1$. This is easy to satisfy as k can be considered as the ratio between the heat conduction in the case the low temperature is equal to zero and the compressor power P. In the case of zero heat conductivity $k = 0$ and $x_{B0} = 1/2$.

Fig. 4 is a plot of the optimum two-stage base temperature

$$t_{B20} = t_{B2}(x_{B0}(k), k) \tag{58}$$

as function of the heat-conduction parameter k. For comparison the corresponding first-stage temperature $t_{B10} = t_{10}(x_{B0}, t_{B20})$ and the single-stage minimum temperature t_B (Eq. (23)) are given as well. These curves show the natural result that the temperatures get lower when the heat conduction get smaller and that the single-stage temperature is in between the first and second-stage temperatures.

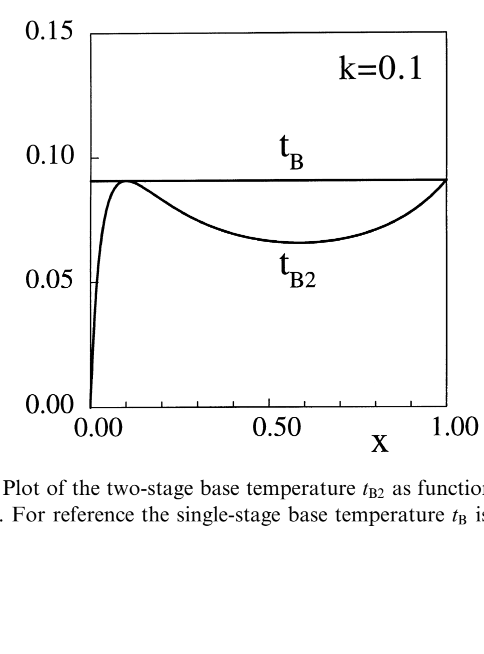

Fig. 3. Plot of the two-stage base temperature t_{B2} as function of x for $k = 0.1$. For reference the single-stage base temperature t_B is given as well.

5. Discussion

Under several simplifying assumptions we have derived analytical expressions which describe what happens when a stage is added to a single-stage PTR. For maximum cooling power the optimum positioning of the first stage, x_0, given by Eq. (54), ranges from 0.5 (for $t_2 = 0$) to 1 (for $t_2 = 0.5$). In engineering terms this means that for very low T_2-values the first stage should be connected to the middle of the regenerator and that, for temperatures $T_2 > 0.5T_H \approx 150$ K, adding a stage has no positive effect. Near the maximum the $\psi - x$ curves are fairly flat which means that, in a practical situation, the engineer has some freedom to connect the first stage at a convenient position without too much loss in efficiency.

If one is interested in reaching the lowest possible temperatures the connection point of the first stage varies between the mid ($x_{B0} = 1/2$) and the cold end ($x = 1$) of the regenerator (Eq. (57)). In the case of small

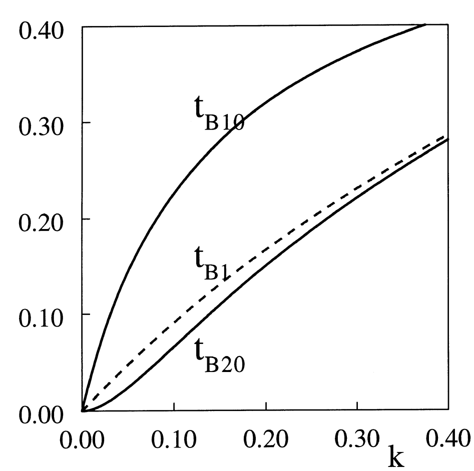

Fig. 4. Plots of the single-stage temperature t_{B1} (dotted line) and the first and second stage temperatures t_{B10} and t_{B20} (full curves) of a two-stage PTR in the optimum situation as functions of the heat-conduction parameter k.

heat conductivity the best position is at the mid point of the regenerator.

For a proper interpretation of our results, it is emphasized that our calculations apply to a highly idealized situation. However, in cases where dissipation mechanisms, other than heat conduction, are dominant a similar approach should be applied. It is a challenge to investigate whether it is possible to derive analytical expressions for more general cases, e.g., with flow resistance or bad heat contact in the regenerator, but we expect that this may turn out to be impossible. In that case a numerical approach would be unavoidable.

References

[1] de Waele ATAM, Steijaert PP, Gijzen J. Thermodynamical aspects of pulse tubes. Cryogenics 1997;37:313.
[2] de Waele ATAM, Steijaert PP, Koning JJ. Thermodynamical aspects of pulse tubes II. Cryogenics 1998;38:329.