A brief, incomplete, and possibly incorrect introduction to statistics
or
A guide to physicists' jargon

The terms I hope to define

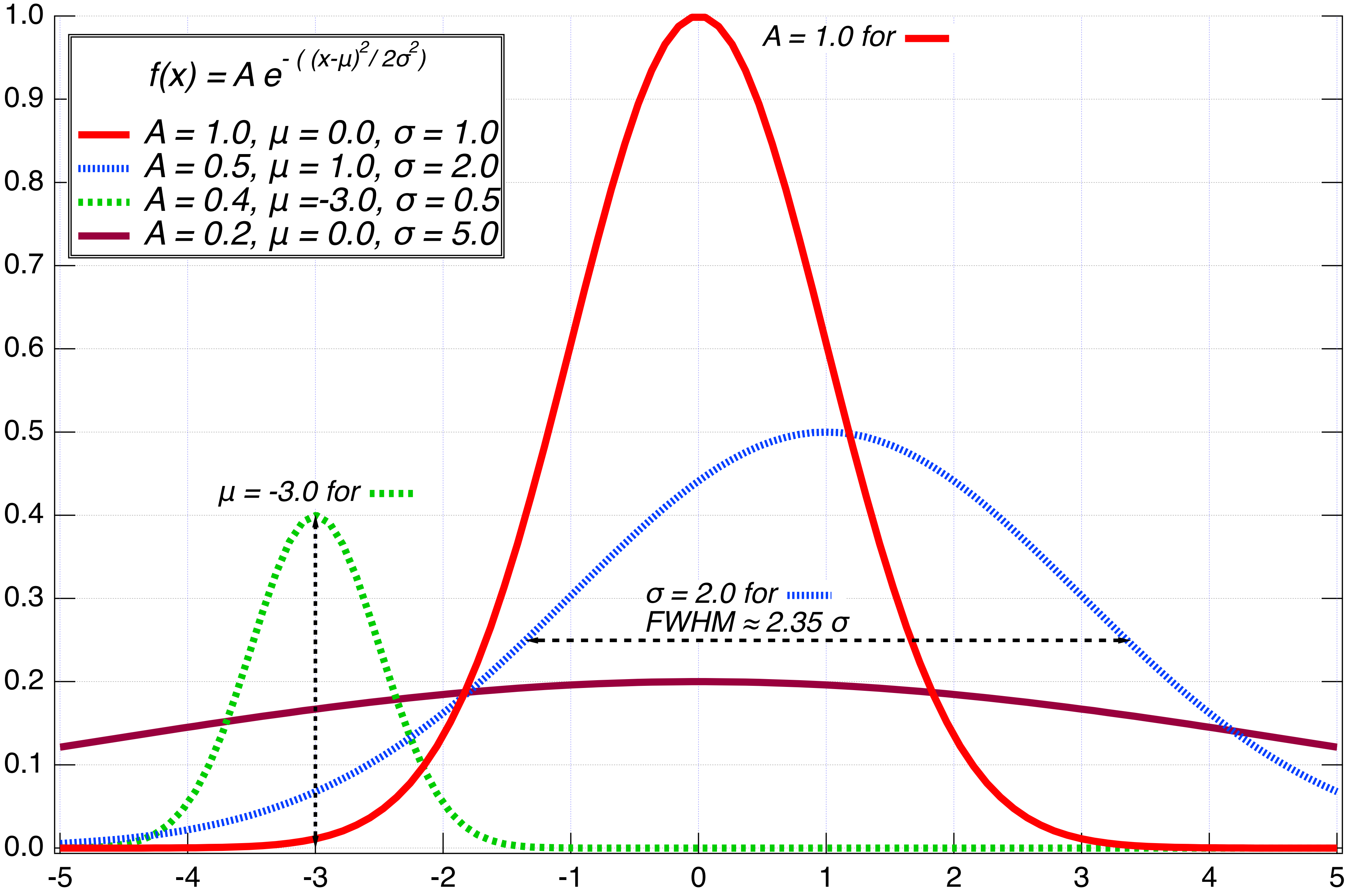
- gaussian
- chi-squared
- chi-squared per degrees of freedom
- sigma (as in "we're looking for a five-sigma effect")
- systematic error

Professor Michael Shaevitz, Director of Nevis Labs, is our expert on statistics. I'm half-remembering what he taught me, and making up the rest.

gaussian

$$f(x) = e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

- μ is the mean of the distribution.
- σ = the standard deviation; it’s related to the “full width at half maximum” (FWHM) of the curve by FWHM = $2\sqrt{2\ln 2}\sigma \approx 2.35\sigma$.
- e = Euler’s constant, a transcendental number that occurs often in calculations that relate to growth and increase. It’s formally defined as $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$.


$f(x) = A\, e^{-((x-\mu)^2/2\sigma^2)}$
A = 1.0, μ = 0.0, σ = 1.0
A = 0.5, μ = 1.0, σ = 2.0
A = 0.4, μ =-3.0, σ = 0.5
A = 0.2, μ = 0.0, σ = 5.0
A = 1.0 for
μ = -3.0 for
σ = 2.0 for
FWHM ≈ 2.35 σ
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
-5
-4
-3
-2
-1
0
1
2
3
4
5

caveats

- If you want to work with the normal distribution as a “probability density function” then you’ll want to include a normalization so the integral $\int_{-\infty}^{\infty} \mathcal{N}(x)dx = 1$

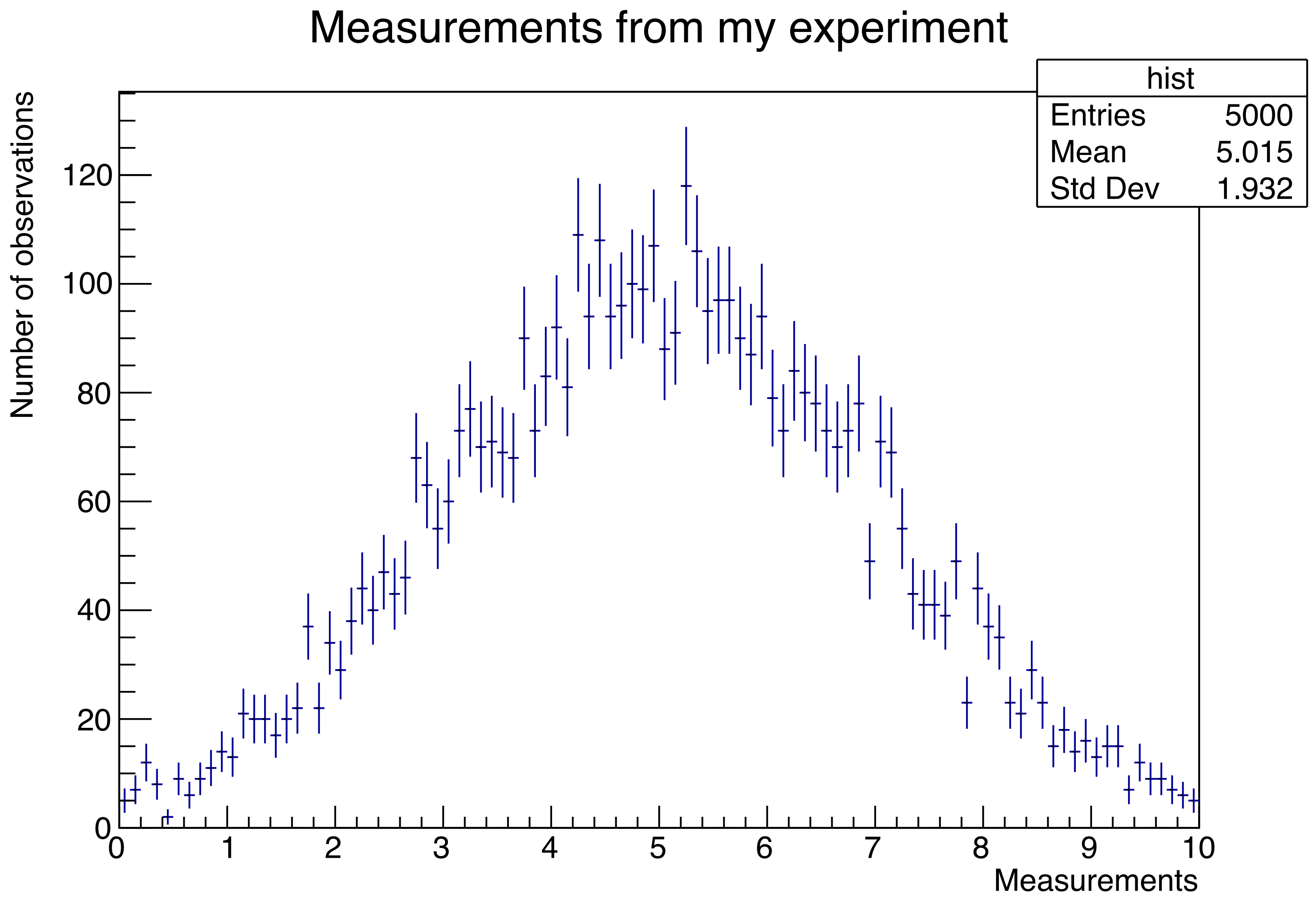
$$N(x\colon \mu, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

However, if you work with this form you can’t fiddle with the amplitude of the distribution. You don’t usually see it in physics.

- There are other functional forms in physics than the gaussian! However, I’m lazy, so that’s the only one I use in the tutorial.

- It makes some sense to stick with gaussians, since the sum of many random processes (even non-gaussian ones) tends towards a gaussian.

chi-squared



What’s the probability that the underlying distribution of this histogram is a gaussian?

chi-square for a 1D histogram

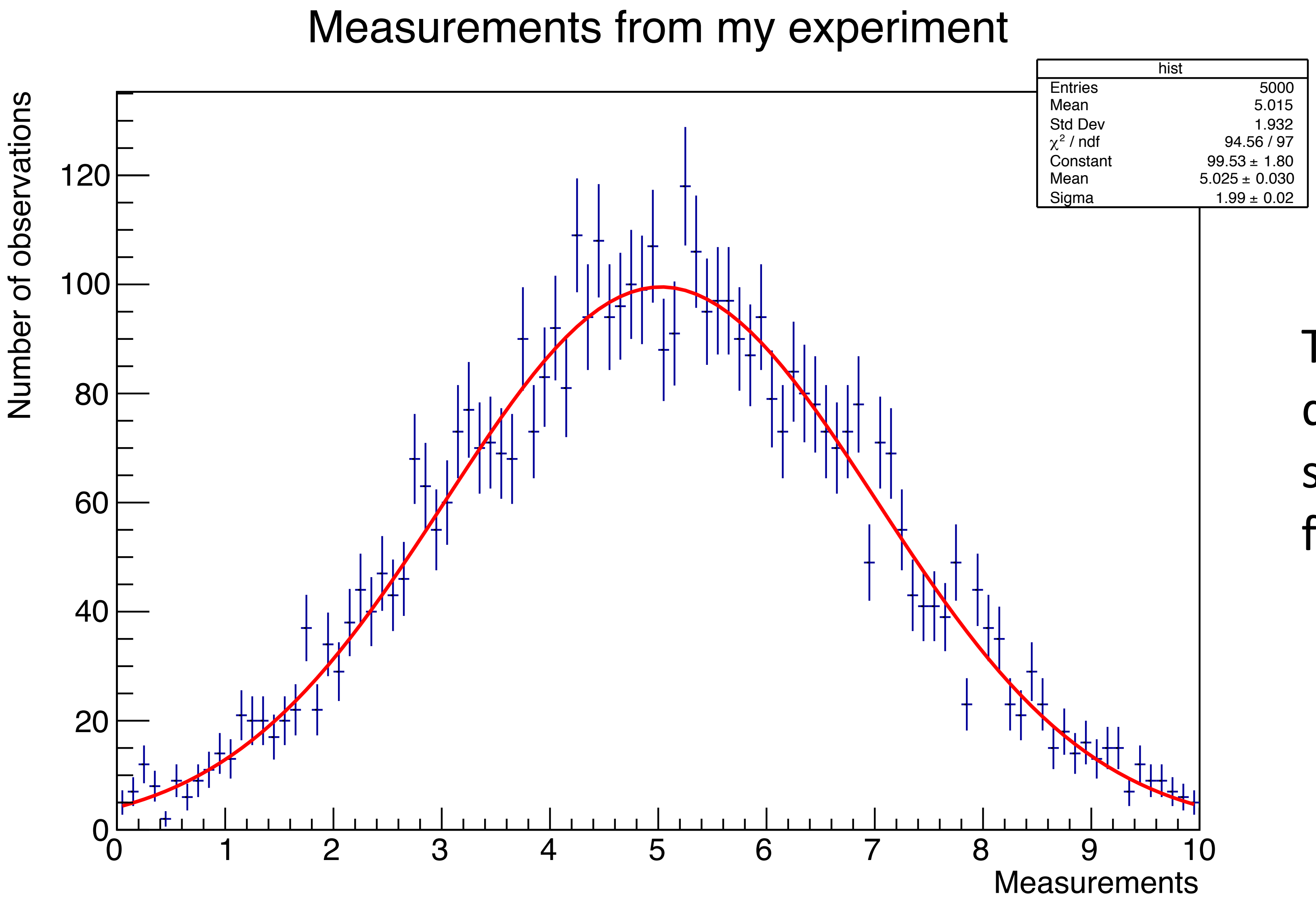
$$\chi^2 = \sum_i \frac{\left(y_i - f\left(x_i; p_j\right)\right)^2}{e_i^2}$$

where:

- *i* means the *i*-th bin of the histogram (more generally, the *i*-th data point you've gathered).
- y_i means the data (or value of) the *i*-th bin of the histogram.
- e_i means the error in the *i*-th bin of the histogram (i.e., the size of the error bars).
- $f\left(x_i; p_j\right)$ means to compute the value of the function at x_i (the value on the *x*-axis of the center of bin *i*) given some assumed values of the parameters $p_0, p_1, p_2 \ldots p_j$.

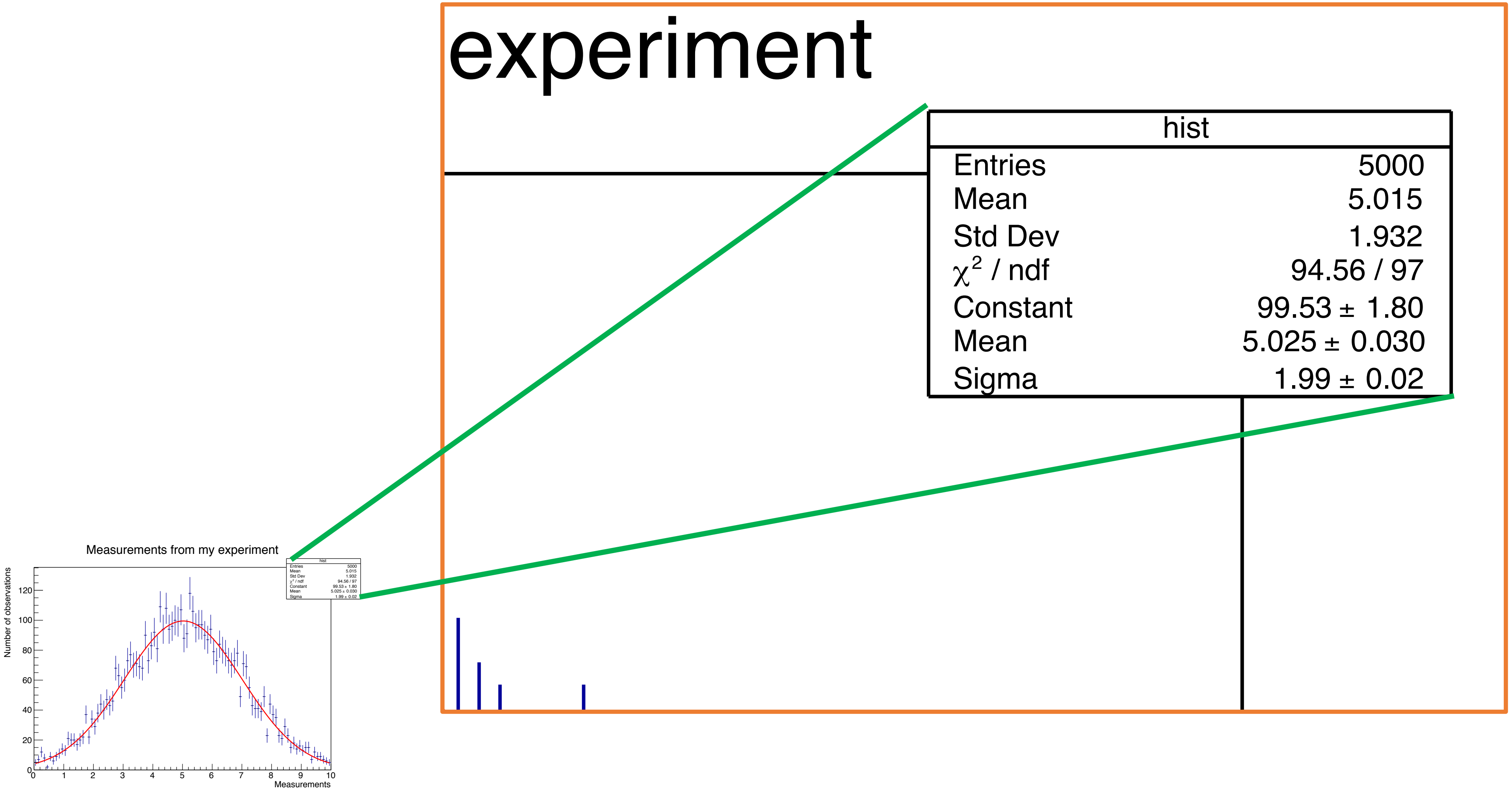
The process of "fitting" means to test different values of the parameters until you find those that minimize the value of χ^2.

Fit ≡ Test different values of A, μ, and σ until you find those that minimize the value of χ^2.



The underlying program that does this is Minuit. It’s been a standard program for finding function minima for decades.

chi-squared per (number of) degrees of freedom



hist	
Entries	5000
Mean	5.015
Std Dev	1.932
χ^2 / ndf	94.56 / 97
Constant	99.53 ± 1.80
Mean	5.025 ± 0.030
Sigma	1.99 ± 0.02

What value of chi-square do you expect from a fit?

$$\chi^2 = \sum_i \frac{\left(y_i - f\left(x_i; p_j\right)\right)^2}{e_i^2}$$

- For each individual bin *i* the data forms a little gaussian distribution of its own with a mean of y_i.
- The e_i acts as a scale of the difference between y_i and the function *f(x)*. So if *f(x)* is a reasonable approximation to y_i, $(y_i\text{-}f(x))/e_i$ will be around ±1.
- You add up those “1”s for each of the bins, and you might anticipate that χ^2will be roughly equal to *i*, the number of bins.

But we have to adjust that…

There are three "free parameters" in the fit: A, μ, σ.

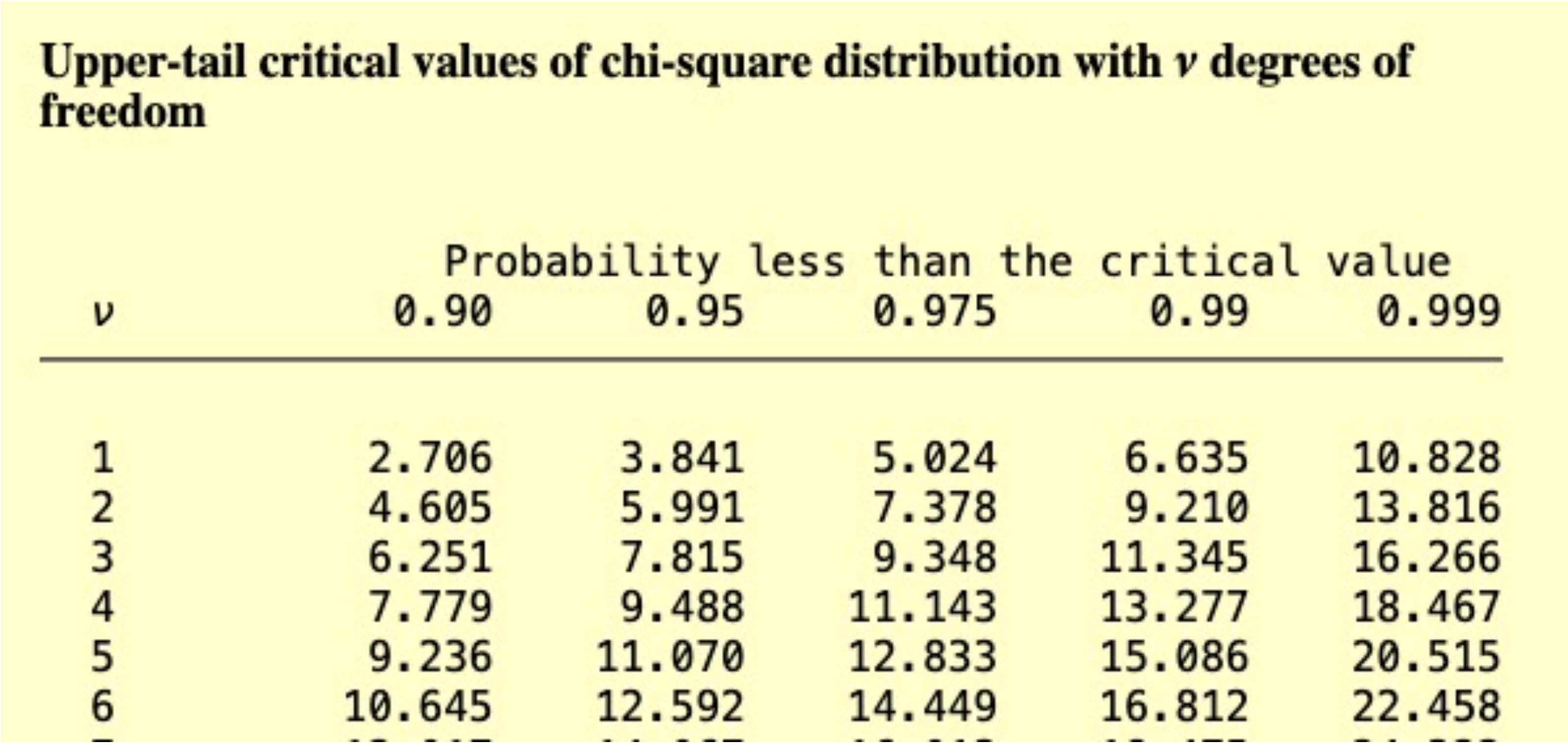
They're going to be varied to make the chi-squared smaller. The net effect is that total number of "degrees of freedom" is:

DOF = number of data points
– number of free parameters in the function

In that fit a couple of pages ago, χ^2 / ndf = 94.56 / 97.

What does this tell us?

Look it up on the web



Upper-tail critical values of chi-square distribution with ν degrees of freedom

	Probability less than the critical value				
ν	0.90	0.95	0.975	0.99	0.999
1	2.706	3.841	5.024	6.635	10.828
2	4.605	5.991	7.378	9.210	13.816
3	6.251	7.815	9.348	11.345	16.266
4	7.779	9.488	11.143	13.277	18.467
5	9.236	11.070	12.833	15.086	20.515
6	10.645	12.592	14.449	16.812	22.458

For the typical fits that we do in physics (lots of data points), it’s sufficient that χ^2 / ndf is roughly 1.

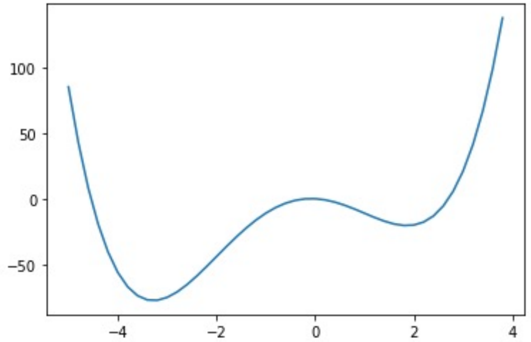
Why might χ^2 / ndf be much greater than 1?

- There's something wrong in the routine that's calculating χ^2

- The model that's being assumed for the function does not have enough parameters.

- The error bars for your data are too small

- Function-minimization programs can get "stuck" in a local minimum that's not the actual true minimum



Why might χ^2 / ndf be much less than 1?

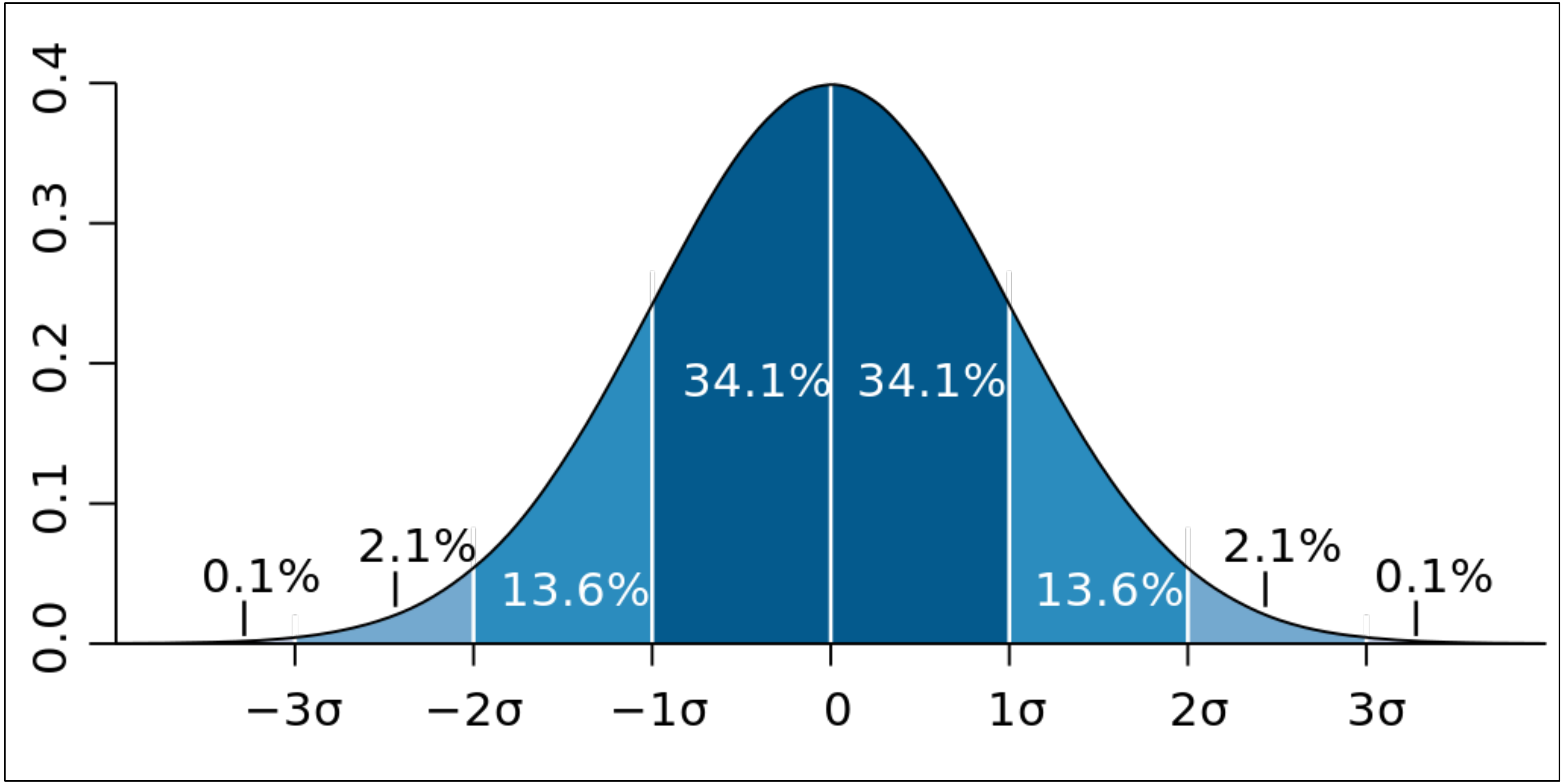
- Again, something wrong in the χ^2 calculation.

- Too many free parameters in the function you’re using to fit.

- The errors on your data are too large.

- Someone has gone wrong in your data-analysis process and you’re “tuning” the data to the model you want to fit.

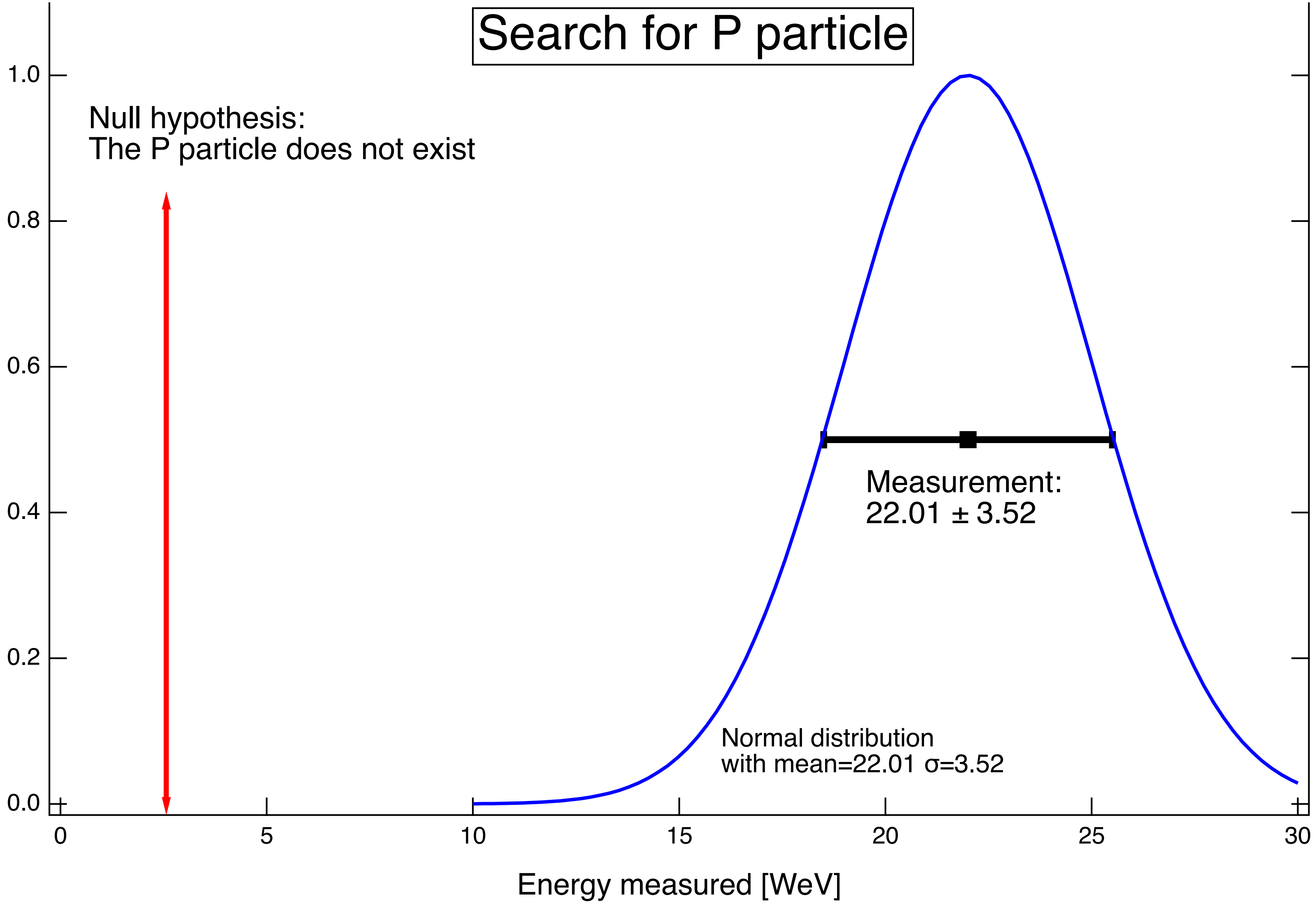
“We’re looking for a five sigma effect”



3σ = “evidence”

5σ = “discovery”

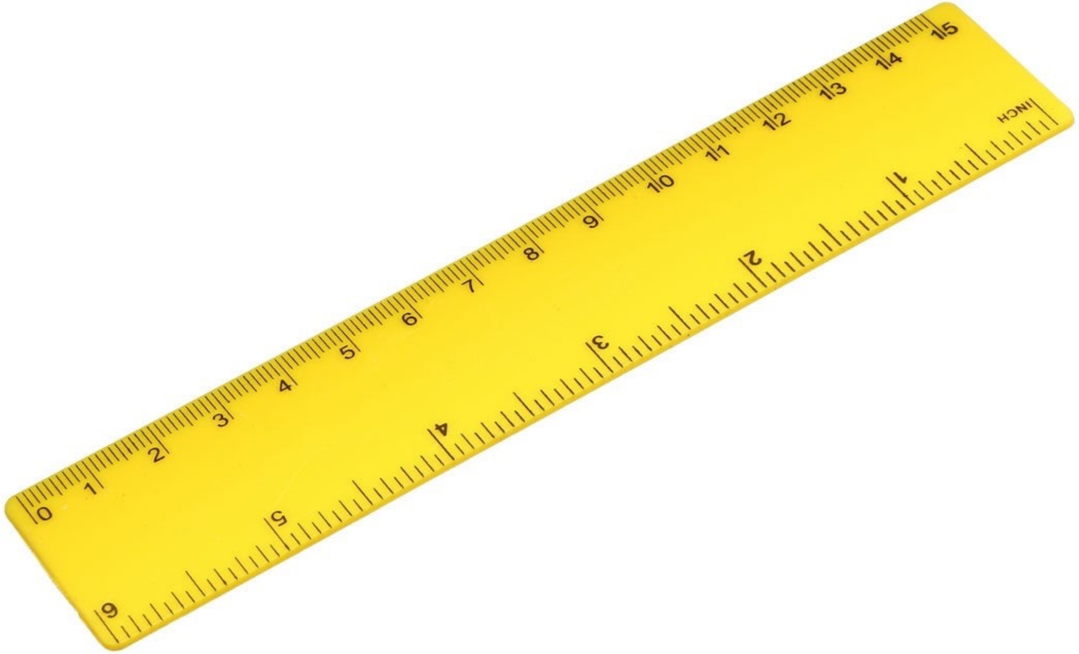
This how I think of it



Systematic error
versus statistical error

Statistical error = random errors in the measurement.

Systematic error = a bias in the measurement, but you don’t know how much that bias is.



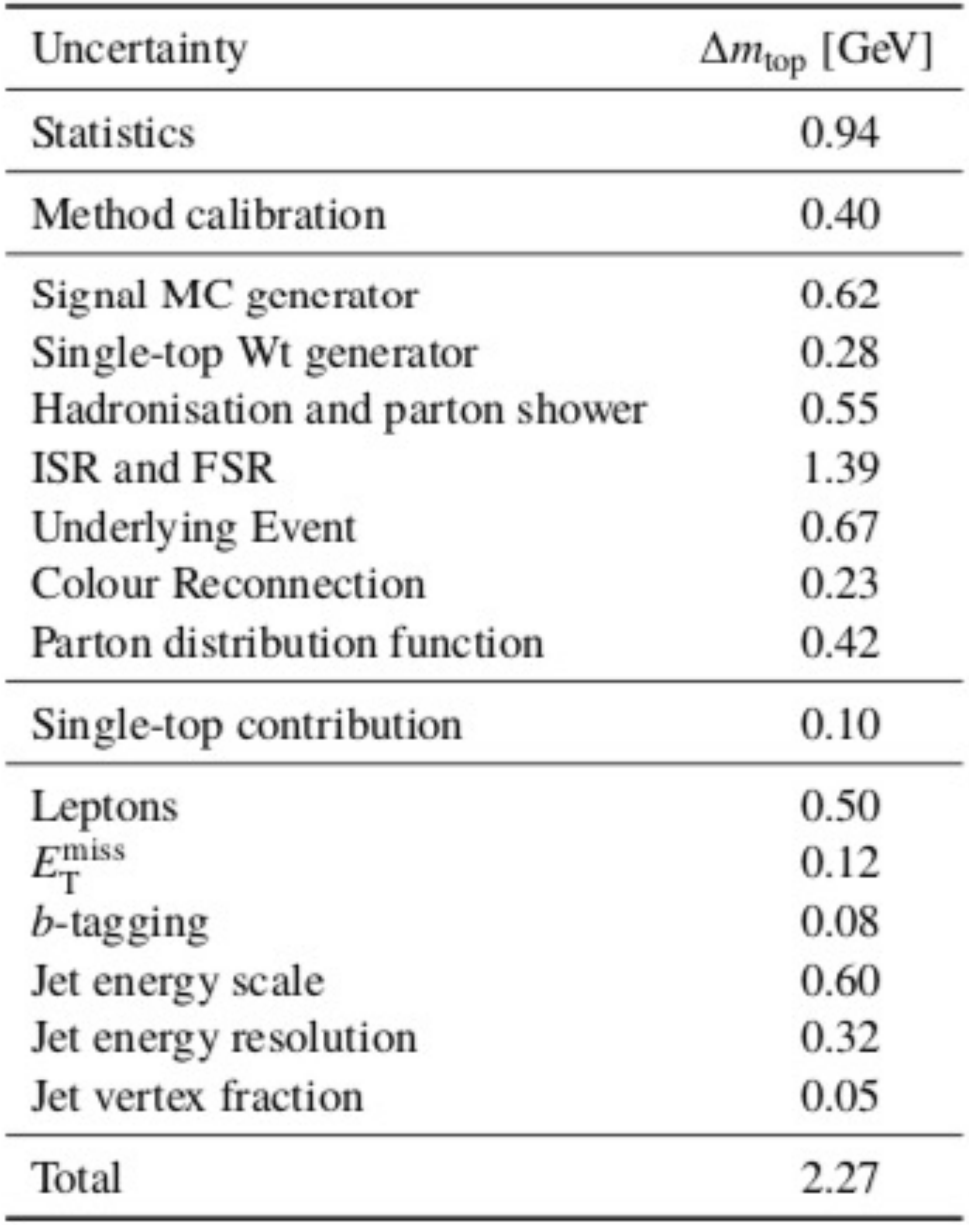
Example of an experiment's systematic errors

ATLAS PUB Note

ATL-PHYS-PUB-2018-001

31st January 2018

Investigation of systematic uncertainties on the measurement of the top-quark mass using lepton transverse momenta



Uncertainty	Δm_{top} [GeV]
Statistics	0.94
Method calibration	0.40
Signal MC generator	0.62
Single-top Wt generator	0.28
Hadronisation and parton shower	0.55
ISR and FSR	1.39
Underlying Event	0.67
Colour Reconnection	0.23
Parton distribution function	0.42
Single-top contribution	0.10
Leptons	0.50
$E_{\mathrm{T}}^{\mathrm{miss}}$	0.12
b-tagging	0.08
Jet energy scale	0.60
Jet energy resolution	0.32
Jet vertex fraction	0.05
Total	2.27