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3rd lecture

# "THERMODYNAMICS OF STRONG INTERACTIONS"

by

R. Hagedorn

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## 3.5 Relation of $\sigma(E, V)$ to phase space

Consider a gas of N free particles of mass m and spin zero in a cubical box of volume $V = L^3$. To calculate the number of levels in {E, dE}, we start by writing down a single particle wave function. As the particle is enclosed in a box, not all momenta $\vec{p}$ are permitted. The simplest way to obtain the permitted ones ($\vec{p}_\alpha$), is to imagine the box repeated periodically in all three space directions, and then to ask for the set of plane wave states having the same periodicity. Thus if

$$\vec{n} = (n_1, n_2, n_3) \quad \text{with} \quad n_i = 0, \pm 1, \pm 2 \ldots$$

then $$e^{i \vec{p}_\alpha \vec{x}} = e^{i[\vec{p}_\alpha (\vec{x} + \vec{n} L)]} \tag{3.32}$$

Hence $$L \vec{p}_\alpha \vec{n} = k \cdot 2\pi \quad ; \quad k = 0, \pm 1, \pm 2 \ldots$$

therefore the vector $\vec{\alpha} = (L/2\pi)\, \vec{p}_\alpha$ can only have integral positive and negative components:

$$\vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3) \quad ; \quad \alpha_i = 0, \pm 1, \pm 2 \ldots$$

The number of permitted states equals the number of lattice points $\vec{\alpha}$. As the $\vec{\alpha}$'s sit on a unit lattice, the number of $\vec{\alpha}$-points equals the volume of $\vec{\alpha}$-space:

$$[\text{number of states in } d^3\alpha] = d^3\alpha = \left(\frac{L}{2\pi}\right)^3 d^3p$$

in other words: the number of states in $d^3p$ is

$$z(\vec{p})\, d^3p = \frac{V}{(2\pi)^3}\, d^3p = \frac{V}{2\pi^2}\, p^2 dp \tag{3.33}$$

The same result is often "derived" from the uncertainty relation $\Delta x \cdot \Delta p > h$ which says that for given $\Delta x = L$, only states with $\Delta p \gtrsim h/L$ can be distinguished. Therefore, in dp, at most, $dp/\Delta p = Ldp/h$ different states can be counted. Extending this argument to three dimensions yields

$$z(\vec{p})\, d^3p = \left(\frac{L}{\hbar}\right)^3 d^3p$$

which is identical to (3.33), because of our convention $\hbar = h/2\pi = 1$.

We now derive $\sigma_N(E, V)$ for N particles. Obviously, since the particles do not interact, the differential number of states for N particles is particles is

$$\left[z(\vec{p})\, d^3p\right]^N = \left[\frac{V}{(2\pi)^3}\right]^N \prod_{i=1}^{N} d^3p_i$$

and the number of states $\sigma(E, V)$ is obtained by integrating over all momenta with the condition that $\sum\sqrt{p_i^2 + m^2} = E$; hence for identical particles

$$\sigma_N(E, V) = \frac{V^N (2\pi)^{-3N}}{N!} \int \delta\left(E - \sum_{i=1}^{N} \sqrt{p_i^2 + m^2}\right) \prod_{i=1}^{N} d^3p_i \qquad (3.34)$$

This is a familiar expression, which -- apart from a minor difference -- looks like the probability for an N-particle state in the Fermi statistical model (Fermi 1950): there is an "interaction volume" $(V/h^3)$ taken to the $N^{th}$ power, there is the N! for equal particles and there is the phase space (more precisely: momentum space) integral which is so dear to us in particle physics.

The minor difference is that the $\delta$-function for momentum conservation, which we are used to in particle physics, is missing. If it were not a minor difference, thermodynamics of strong interactions would be more complicated. That it <u>is</u> a minor difference, is exactly due to the circumstance mentioned at the end of the last section, namely that $\ln \sigma(E, V)$ and not $\sigma(E, V)$ itself is the physically interesting quantity. As this

point is important, we illustrate it by calculating both densities, with and without momentum conservation, for the case of non-relativistic particles, where the phase space integrals can be solved exactly. As the factor $V^N \cdot \left[N!(2\pi)^{3N}\right]^{-1}$ is the same in both cases, we compare the phase space integrals. In the case of momentum conservation we should not require the total momentum to be exactly zero (then the integral vanishes), but only that the total momentum $\vec{P} = \sum \vec{p}_i$ be limited to $|\vec{P}| < \Delta P$. Then we compare two integrals of equal dimension (we put $E - N \cdot m = E_{kin}$):

$$I(E,N) = \int \delta\left(E_{kin} - \frac{1}{2m}\sum p_i^2\right) \prod_i d^3p_i \tag{3.35}$$

$$I(E,N,\Delta P) = \left[\frac{4\pi}{3}\Delta P^3\right] \int \delta\left(E_{kin} - \frac{1}{2m}\sum p_i^2\right) \delta^3\left(\sum \vec{p}_i\right) \prod_i d^3p_i \tag{3.36}$$

The result of some lengthy algebra is (without approximations)

$$I(E,N) = \frac{\left(2\pi m E_{kin}\right)^{\frac{3N}{2}}}{E_{kin}\,\Gamma\left(\frac{3N}{2}\right)} \tag{3.35a}$$

$$I(E,N,\Delta P) = \frac{\left(2\pi m E_{kin}\right)^{\frac{3N}{2}}}{E_{kin}^{5/2}\,\Gamma\left(\frac{3N-3}{2}\right)} \cdot \frac{1}{\left(2\pi m N\right)^{3/2}} \cdot \left[\frac{4\pi}{3}\Delta P^3\right] \tag{3.36a}$$

Hence, with some slight approximation (Sterling formula)

$$\frac{\sigma_N(E,V)}{\sigma_N(E,V,\Delta P)} = \left[\frac{E_{kin}}{\Delta P^2/2m}\right]^{3/2} \tag{3.37}$$

which diverges for $E_{kin} \to \infty$. On the other hand,

$$\frac{\ln \sigma_N(E,V)}{\ln \sigma_N(E,V,\Delta P)} \Rightarrow 1 \tag{3.38}$$

no matter how N and/or $E_{kin}$ tend to $\infty$ . Thus, thermodynamically it is a unnecessary luxury to keep the total momentum fixed (within $\Delta P$). In fact, eq. (3.37) tells us clearly why. Let us ask: how large a $\Delta P$ must we allow in order to make not only (3.38) but even (3.37) equal to one? The answer is

$$\frac{\Delta P^2}{2m} = E_{kin} = N \cdot \frac{\overline{p^2}}{2m} \qquad \text{hence}$$

$$\Delta P = \sqrt{N} \cdot \sqrt{\overline{p^2}} \tag{3.39}$$

This is an obvious result: if we add up N random vectors with mean length $\sqrt{\overline{p^2}}$, then the result is just given by the r.h.s. of this equation. In other words:

> $\sigma(E, V, \Delta P)$ and $\sigma(E, V)$ differ only then from each other, if we require the total momentum $\Delta P$ to be much smaller than its statistical expectation value $\sqrt{\overline{p^2} \cdot N}$ .

For a sufficiently large number of particles, this value is, however, ridiculously small. The mean collective velocity $\Delta v$ and kinetic energy $\Delta E$ connected to it is given by $\Delta P = M \cdot \Delta v = N \cdot m \cdot \Delta v$, and from (3.39) it then follows

$$\Delta v = \sqrt{\overline{v^2}/N} \quad ; \quad \Delta E_{kin} = \frac{\Delta P^2}{2Nm} = \frac{E_{kin}}{N} = \frac{3}{2} T \tag{3.40}$$

In other words: if we neglect momentum conservation in the space integral, then we permit random collective movements of the system which are by a factor $\sqrt{N}$ smaller than the mean movement per particle, and the total kinetic energy contained in this collective motion is of the same order as the kinetic energy per particle and equals the thermic energy at temperature T of the whole system considered as a mass point.

Of course, the thermodynamics of a macroscopic system is insensitive to whether we fix it by bolts and screws to the floor or just put it there unfastened*). This is the physical meaning of eq. (3.38). Needless to show that a similar result is obtained if one considers conservation of angular momentum (Cerulus, 1961; Ericson, 1961; Joos and Satz, 1964; Koba, 1960; Kotanski and Zalewski, 1965; Zalewski, 1965). These considerations lead us to a few interesting observations for the application in particle physics.

## 3.6 The bridge between particle physics and thermodynamics

- The bridge from particle physics to thermodynamics is the phase space integral. As far as a thermodynamic approach is justified, momentum conservation is unimportant. If necessary, we may reinforce it on final results.

- With $S = \ln \sigma(E, V)$ and $\partial S/\partial E = 1/T$ [see (3.14)], we can assign a temperature to any phase space integral with the result (non-relativistic particles $m >> T$)

$$\frac{3}{2}T = \begin{cases} E/(N-2/2) \approx E/N & \text{(no momentum conservation)} \\ E/(N-5/3) \approx E/N & \text{(with momentum conservation)} \end{cases}$$

Again, this does not depend significantly on momentum conservation.

Of course: the assignment of a temperature to a given N-particle state with given energy E makes sense only, if we know that the phase space is isotropically populated -- in other words: if collective motions have been separated off, as we did in section 1.

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*) Total momentum is conserved for the unfastened system!

- Our way from the single particle wave function to the phase space integral has led us to the old, non-covariant momentum space integral. Of course, the whole expression (3.34) for $\sigma_N(E, V)$ is covariant. If one would insist on making the momentum space itself covariant by replacing $d^3p_i$ by $d^3p_i/2E_i$, then one has, at the same time, to replace V by $V \cdot 2E_i$ (and leave it under the integral). This is equivalent to normalizing the wave function of particle i in $V \cdot 2E_i$ instead of in V. In particle physics this has the welcome consequence that the S-matrix elements become invariant, but for a thermodynamic description it is inconvenient to give to each particle another volume that depends on the particle's energy.

- In order to give a thermodynamic description of hadronic processes, we shall thus start from an S-matrix formulation with non-covariant phase space. As the particle number is not fixed (we do <u>not</u> describe single events but averages over many), we have to sum over $N = 0 \ldots \infty$. As furthermore we shall take into account all kinds of hadrons, we have sum over the hadronic mass spectrum. Thus our number of states will be something like

$$\sigma(E, V) = \sum_N \sum_{\text{masses}} \sigma_N(E, \text{masses}, V)$$

where $V^N$ will correspond to an averaged squared matrix element [compare (3.34)] and $\sigma_N$ to an N-particle phase space integral. Such integrals are difficult to calculate, and the sum over all N and all kinds of particles would require the evaluations of millions of them. Remembering eq. (3.27):

$$Z(V, T) = \sum_i e^{-\frac{E_i}{T}} = \int \sigma(E, V) e^{-\frac{E}{T}} dE \quad ,$$

we see, that we have the choice between the calculation of millions of phase space integrals (microcanonical description, E fixed) or of the sum over $\exp(-E_i/T)$ (canonical description, T fixed). It will turn out that while the former is literally impossible, the latter is straightforward.

- Note again, that the whole makes sense only after we have separated off collective motions (Chapter 1).

## 4. MOTIVATION OF THE PARTICULAR STRUCTURE OF HADRON THERMODYNAMICS

In this section, we motivate (we do not prove) the particular hadron thermodynamics which claims to be able to take into account strong interactions -- maybe fully; certainly the main part of them -- in spite of the well-known difficulties in the theory of real gases.

### 4.1 We start from S-matrix

In chapter 1, we have separated off collective motions. We therefore deal with excited hadronic matter in the rest frame of a co-moving observer who, according to Postulate 1, sees only isotropic motion in his immediate neighbourhood. Such a situation arises e.g. in $p\bar{p}$ annihilation at rest. Suppose we could somehow prepare an initial state $|i>$ of high energy with zero baryon number, isospin and strangeness and with negligible momentum and angular momentum. The probability to observe N final particles is

$$P(E,N) = \int |<f|S'|i>|^2 \delta(E-\Sigma E_i)\,\delta^3(\Sigma \vec{p}_i) \prod_i^N d^3p_i$$

$$\equiv \int |S'|^2 dR_N(E, m_1, m_2, \ldots m_N) \qquad (4.1)$$

(The second expression defines a shorthand notation.) We use the non-covariant $d^3p$ for the reasons explained in the last section; states, between which the reduced S-matrix (into which we absorb all factors like 1/N!) is taken, are then normalized over a fixed volume V. By hypothesis, the initial state is isotropic, hence, the phase space is isotropically populated. If we knew the squared matrix element, then we could take its mean value in front of the integral and write

$$P(E,N) = \overline{|S'|^2} \cdot \sigma(E,N) \qquad (4.2)$$

Unfortunately, this would not yet permit us to go over to thermodynamics, because $\overline{|<S'>|^2}$ itself is a function of E and N. We do not wish to make the naive assumptions of the statistical model; we rather wish to take interaction seriously.

4.2 The method of Beth and Uhlenbeck and of Belenkij

The idea of Beth and Uhlenbeck is simply this : $\sigma_N(E, V, m_1 \ldots m_N)$ is the level density of N non-interacting particles of masses $m_1 \ldots m_N$. Suppose now two of them, 1 and 2, rigidly bound together, then we have N - 1 non-interacting particles with level density $\sigma_{N-1}(E, V, m_{12}, m_3 \ldots m_N)$, where $m_{12}$ is the mass of the bound state: *the interaction that is responsible for the rigid binding, manifests itself in a change of the level density.*

Take a realistic example: a dilute He gas is perfectly described by assuming N free, non-interacting He atoms, instead of 6 N protons, neutrons, electrons with complicated, unmanagable interactions. Thus, interaction may be represented by a change of the level density. After all, in eq. (4.2), both factors are functions of E and N, and nobody tells us how exactly they factorize, and with what N. Though interaction need not lead to bound states, it is tempting to ask: how does it influence the level density? This is what Beth and Uhlenbeck do.

Consider N particles in a box. This time, we choose it spherically with radius R (the shape of the normalization box does not matter). The single particle wave functions are now spherical waves. From the condition that they vanish on the walls (r = R), one then derives (in a more complicated way) exactly the same level density as in (3.34). Now we single out particles 1 and 2 and ask for the effect of their interaction. If r and p' denote their relative distance and momentum, then their internal asymptotic lth partial wave will be

$$\psi_\ell(r, p') \sim \frac{1}{r} \sin\left(p'r - \ell\frac{\pi}{2} + \delta_\ell\right) \qquad (4.3)$$

The boundary condition at R implies

$$p'R - \ell\frac{\pi}{2} + \delta_\ell(p') = n_\ell \pi$$
$$n_\ell = 0, 1, 2 \ldots \qquad (4.4)$$

and the number of permitted partial wave states in {p', dp'} becomes

$$\frac{dn}{dp'}dp' = \left(\frac{R}{\pi} + \frac{1}{\pi}\frac{d\delta_\ell}{dp'}\right)dp' \qquad (4.5)$$

Thus, the level density of the unperturbed 2-particle state ($dp' \cdot R/\pi$) acquires an additional term, due to the lth partial wave phase shift. We have the choice between two descriptions:

> *either* we use (4.1) with unperturbed phase space and describe interaction by $|S'|^2$,
> *or* we calculate phase space with distorted waves and eliminate the corresponding part of the interaction from $|S'|^2$.

[The most modern formulation of this old idea is due to Dashen, Ma and Bernstein (1969).]

Since the knowledge of all phase shifts in all channels -- including, of course, 2, 3, ... n-body phase shifts -- is equivalent to the full S-matrix, we might attempt to push the method of Beth and Uhlenbeck to the limit, by extending the previous consideration to *all* phaseshifts (between any number of particles and in all partial waves) and to unlimited creation and absorption of particles -- in other words: by calculating all possible phase space integrals with fully distorted wave functions, thereby reducing the S-matrix to an uninteresting constant and expressing the effect of strong interactions entirely by the change of level density

they bring about by phase shifts and particle creation. (As an analogy, remember the improvements in the theory of nuclear beta-decay, if, for the outgoing electron, one uses Coulomb wave functions instead of plane waves).

Unfortunately, in spite of the effort of many physicists, the full S-matrix and with it the set of all phase shifts of strong interactions are unknown. How then can we include the effect of the set of all phase shifts? We do it by a circular syllogism which amounts to a complete and universal bootstrap of all hadrons. A circular reasoning may start anywhere; let us start from the fact that strong interactions produce resonances.

If there exists, in the 1th partial wave at $p' = p^*$, a resonance between particles 1 and 2, then $\delta_1$ increases by $\pi$ in a small interval around $p^*$. Let this resonance have zero width -- we shall relax on this later on -- hence, $(1/\pi)\, d\delta_1/dp' = \delta(p' - p^*)$. Thus, if we insert (4.5) back into (4.1) and sum over partial waves, the first term $R/\pi$ yields the old integral (4.1), while the second one, $\delta(p' - p^*)$, yields an integral with particles 1 and 2 appearing as a single object with mass $m^*$ = invariant mass $M_{12}(p^*)$. Hence,

$$P(E,N) = \int |S''|^2 dR_N(E, m_1, m_2, \ldots m_N) + \int |S''|^2 dR_{N-1}(E, m_1^*, m_3 \ldots m_N) \tag{4.6}$$

where S' is replaced by S'', indicating that part of the interaction is eliminated from S'. This transition from (4.1) to (4.6) with the help of (4.5), when there is a resonance, has been done explicitly for the first time by Belenkij (1956). Since then, it has become a standard technique in the statistical model. However, in all subsequent statistical model applications, only a small number of resonances was permitted, so that the beautiful method of Beth and Uhlenbeck was not really exploited.

The reason was simply, that if one takes into account many resonances, the number of phase space integrals becomes astronomical: at 30 GeV it is about $10^{10}$, if all presently known particles and resonances are taken, but there are many more waiting to be discovered. We, however, shall use thermodynamics, where astronomical numbers are a virtue.

4.3 The momentum spectrum $f_m(\vec{p}, E)$

We wish to calculate the momentum distribution of a given kind of particle (mass m), no matter how many other particles there are, and no matter what they do. For the N-particle state this distribution follows from P(E, N) by suppressing the $d^3p$-integration of the particle considered. For the phase space element with $d^3p$ suppressed, we write $dR_N/d\vec{p}$. Hence, from (4.6)

$$f_m^{(N)}(\vec{p}, E) = \int |S''|^2 \frac{dR_N}{d\vec{p}}(E, u_1, \dots) + \int |S''|^2 \frac{dR_{N-1}}{d\vec{p}}(E, u^*, u_3 \dots) \tag{4.7}$$

Let us continue to eliminate interaction from $|S''|^2$ by introducing resonances between all particles: 2-body, 3-body, ..., n-body resonances and even resonances between particles and resonances, between resonances and resonances and so on.

Suppose we knew *all* resonances and suppose that strong interactions are fully described by the set of all possible production processes, together with the set of all hadrons including all resonances ("resonance saturation",-- we shall relax on this later on), then the final result will be

$$f_m(\vec{p}, E)\, d^3\vec{p} = K \cdot d^3p \cdot \sum_c \int \frac{dR}{d\vec{p}}(E, \text{masses}) \tag{4.8}$$

Here K is, what remains of $|S|^2$ after shifting all interactions into phase space: a simple constant. $\sum_c$ goes over all possible configurations: namely all possible particles and resonances and all possible numbers of them. Nobody can actually calculate this by evaluating individual phase space integrals. But just because it is too complex as it stands, thermodynamics can take over: in terms of statistical thermodynamics (4.8) is the microcanonical description of the momentum distribution in a gas, consisting of undetermined numbers of all kinds of freely created particles; in other words: of hadronic black body radiation with fixed energy and with total momentum $-\vec{p}$, where $\vec{p}$ is the momentum at which $f_m(\vec{p}, E)$ is evaluated. As $|\vec{p}|$ is of the order of the average momentum per particle, (it does not matter if it is even ten times bigger), the phase space integrals are to be evaluated at values of $|\vec{p}|$ which essentially lie in the region $\Delta P$, discussed in section 3.5, eq. (3.39). This enables us to neglect momentum conservation; thus we have to deal with hadronic black body radiation in the usual sense. [Once $f_m(\vec{p}, E)$ has been calculated, a momentum cut-off can be imposed on the result in order to repair the leak.]

We now go over from the microcanonical (E-fixed) to the canonical (T-fixed) description and can at once conclude that $f_m(\vec{p}, E)$ must have the form of a Planck distribution (we shall later derive this result from ln Z):

$$f_m(\vec{p}, T(E))\,d^3p = \text{const.} \left[\exp \frac{\sqrt{p^2+m^2}}{T(E)} \pm 1\right]^{-1} d^3p \qquad (4.9)$$

+ for fermions, - for bosons.

## 4.4 Definition of the hadronic mass spectrum

The task may seem to be settled now -- but it is not. Before this formula can be applied, we have to know what value of T belongs to a given energy density. Thus all our arguments, so far, amount only to having shifted the problem; it now reads:

> calculate the relation between energy density E/V and temperature T for hadronic black body radiation.

This new problem is equivalent to calculating either $\sigma(E, V)$, or the partition function; and that we can do if we know which particles and resonances make up our hadron gas: *we need the full hadronic mass spectrum.*

> To count all particles and resonances, sharp or large, including even a smooth background, we introduce the hadronic mass spectrum $\rho(m)$, such that $\rho(m)dm$ is the number of thermodynamically distinct "particles" in the mass interval $\{m, dm\}$ .

The $2J + 1$ orientations of a spin J particle correspond thermodynamically to $2J + 1$ distinct particles; thus $\rho(m)$ contains a spin-isospin multiplicity factor $(2J + 1)(2I + 1)$ and a further factor 2, if particle $\neq$ anti-particle. Resonances contribute to $\rho(m)$ via $(1/\pi)\, d\delta_l/dp$, and therefore need not be sharp any longer; as $\rho(m)$ is anyway conceived as a continuous function (containing a few $\delta$-bumps for the stable particles), it may also accomodate a smooth background. Let "everything necessary", (whatever that means), be contained in this unknown function: the problem has now become to find out what this $\rho(m)$ is. But assume we knew it; then

$$\sum_{\substack{\text{all} \\ \text{configurations}}} [\ldots] \Longrightarrow \sum_{N=0}^{\infty} \sum_{i=1}^{N} \int [\ldots]\, \rho(m_i)\, dm_i$$

Thus our piece of highly excited hadronic matter -- from now on called a fireball -- is thermodynamically described as hadronic black body radiation, enclosed in a box of volume V, containing undetermined numbers of <u>all</u> kinds of hadrons. The number of different sorts of hadrons with mass in $\{m, dm\}$ is given by $\rho(m)dm$. The most important property of this "gas" is:

> Although this gas consists *formally* of free, non-interacting particles, (almost) all strong interactions are properly taken into account by the hadronic mass spectrum $\rho(m)$ and by the freedom to create and annihilate the particles described by $\rho(m)$.

Note the *fundamental difference* between $\rho(m)$, which counts the possible mass-states of the hadrons making up our fireball, and $\sigma(E, V)$, which counts the number of states of the fireball as a whole. Example: in ordinary electromagnetic (em) black body radiation we have $\rho_{em}(m) = 2\delta(m)$, while $\sigma_{em}(E, V) \sim \exp E^{3/4}$. Thus $\rho_{em}$ is zero, except at $m = 0$, while $\sigma_{em}$ grows almost exponentially. The hadronic $\rho(m)$ is of course much richer than $\rho_{em}$, since there are so many particles and resonances; but the real surprise will be that for strong interactions the two "fundamentally different" functions $\rho(m)$ and $\sigma(E, V)$ will become the same!