ACADEMIC TRAINING No. 51

CERN LIBRARIES, GENEVA

AT00000750

5th lecture

"THERMODYNAMICS OF STRONG INTERACTIONS"

by

R. Hagedorn

Lectures given in the
Academic Training Programme of CERN

1970-1971

5. THE PARTITION FUNCTION AND THE SOLUTION OF THE BOOTSTRAP CONDITION

We start from eq. (3.27) which we repeat here:

$$Z(V,T) = \sum_i e^{-\frac{E_i}{T}} = \int_0^\infty \sigma(E,V)\, e^{-\frac{E}{T}}\, dE \qquad (5.1)$$

It is extremely difficult to calculate $\sigma(E, V)$ directly from phase space; surprisingly enough, the first expression, the sum, can be calculated easily. We shall thus do this and then use the identity with the second formulation to impose the bootstrap condition.

In order to compute $\sum \exp(-E_i/T)$, we have to calculate the energies of all possible states, containing any number of any sort of hadrons. However, since all interaction is contained in the mass spectrum $\rho(m)$, *formally* we have free, non-interacting particles. Therefore, a general state ψ_i is the direct product of single particle plane wave states, and its energy E_i is simply the sum of the energies of the individual free particles. Interaction energies do not occur, they are hidden in $\rho(m)$.

5.1 The partition function

In a box of volume V only a discrete set $\{\vec{p}_\alpha\}$ of plane waves is allowed for each particle. Assume for the moment also that the set of all hadrons is discrete, call it $\{m_\gamma\}$. Then the energy of particle γ in the momentum state α is

$$\varepsilon_{\alpha\gamma} = \sqrt{p_\alpha^2 + m_\gamma^2} \qquad (5.2)$$

Let further

$$\nu_{\alpha\gamma} = \begin{cases} 0, 1 & \text{for fermions} \\ 0, 1, 2, \ldots \infty & \text{for bosons} \end{cases} \qquad (5.3)$$

be the "occupation numbers", indicating how many particles of the kind γ have momentum $\vec{p}_\alpha$. Note that here the Pauli exclusion principle -- or: the difference between Fermi-Dirac and Bose-Einstein statistics -- is fully contained in (5.3), whereas in phase space it is difficult to accomodate.

In this notation each state ψ_i is determined (up to a phase) by one complete set of occupation numbers $\{\nu_{\alpha\gamma}\}_i$, which is an $\infty \times \infty$ matrix with elements $\nu_{\alpha\gamma}$, hence $\psi_i \equiv \{\nu_{\alpha\gamma}\}_i$. Thus

$$E_i = \sum_{\alpha,\gamma}^{\infty} \nu_{\alpha\gamma}\, \varepsilon_{\alpha\gamma} \tag{5.4}$$

and

$$Z(V,T) = \sum_{\{\nu\}} \exp\left[-\frac{1}{T}\sum_{\alpha\gamma}^{\infty} \nu_{\alpha\gamma}\,\varepsilon_{\alpha\gamma}\right] \tag{5.5}$$

where this sum goes over all matrices $\{\nu\}$, each representing one state. We put

$$x_{\alpha\gamma} \equiv e^{-\frac{\varepsilon_{\alpha\gamma}}{T}} \qquad \text{(always } < 1\text{)} \tag{5.6}$$

and proceed:

$$Z(V,T) = \sum_{\{\nu\}} \prod_{\alpha\gamma} x_{\alpha\gamma}^{\nu_{\alpha\gamma}} \equiv \prod_{\alpha\gamma}\left[\sum_{\nu_{\alpha\gamma}} x_{\alpha\gamma}^{\nu_{\alpha\gamma}}\right] \tag{5.7}$$

That these two expressions are identical is easily seen by considering the simplest possible example, namely where $\{\nu\}$ is a 2×2 matrix. We split the label γ into two: β for bosons and ϕ for fermions;

thus $\nu_{\alpha\beta} = 0, 1, 2 \ldots \infty$, while $\nu_{\alpha\phi} = 0,1$ only. Then

$$Z(V,T) = \prod_{\alpha\beta} \frac{1}{1-X_{\alpha\beta}} \prod_{\alpha\phi}(1+X_{\alpha\phi}) \tag{5.8}$$

$$\ln Z = -\sum_{\alpha\beta} \ln(1-X_{\alpha\beta}) + \sum_{\alpha\phi} \ln(1+X_{\alpha\phi}) \tag{5.9}$$

Now we go over to the continuum [see (3.33)]

$$\sum_{\alpha}[\ldots] \Longrightarrow \frac{V}{2\pi^2}\int_0^\infty [\ldots] p^2 dp \tag{5.10}$$

$$\sum_{\beta,\phi}[\ldots] \Longrightarrow \int_0^\infty \rho_{B,F}(m)[\ldots]dm \tag{5.11}$$

Thus

$$\ln Z = \frac{V}{2\pi^2}\int_0^\infty p^2 dp \left[\int_0^\infty \rho_F(m)\ln(1+X_{pm})dm - \int_0^\infty \rho_B(m)\ln(1-X_{pm})dm\right] \tag{5.12}$$

We expand the logarithms and obtain

$$\ln Z = \frac{V}{2\pi^2}\sum_{n=1}^\infty \frac{1}{n}\int_0^\infty dm\,dp\,p^2\,\rho(m;n)\,X_{pm}^n \tag{5.13}$$

where

$$X_{pm}^n \equiv \exp\left(-\frac{n}{T}\sqrt{p^2+m^2}\right) \tag{5.14}$$

and

$$\rho(m;n) \equiv \rho_B(m) - (-1)^n \rho_F(m) \tag{5.15}$$

In order to carry out the p-integration, we use the formulae

$$\int_0^\infty p^{2n} \exp\left(-\frac{\sqrt{p^2+m^2}}{T}\right) dp = (2n-1)!!\, m^{2n+1} \left(\frac{T}{m}\right)^n K_{n+1}\left(\frac{m}{T}\right)$$

$$\int_0^\infty p^{2n+1} \exp\left(-\frac{\sqrt{p^2+m^2}}{T}\right) dp = 2^n n! \sqrt{\frac{2}{\pi}}\, m^{2n+2} \left(\frac{T}{m}\right)^{n+1/2} K_{n+3/2}\left(\frac{m}{T}\right)$$

$$(2k-1)!! = 1 \cdot 3 \cdot 5 \cdot \cdots (2k-1) \tag{5.16}$$

where K is the second modified Hankel function. For half-integer index one has

$$K_{n+\frac{1}{2}}(x) = \sqrt{\frac{\pi}{2x}}\, e^{-x} \sum_{k=0}^{n} \frac{(n+k)!}{k!\,(n-k)!} \left(\frac{1}{2x}\right)^k \tag{5.17}$$

which is an elementary function, while for integer n > 1, it is non-elementary with the limiting ($x \to 0, \infty$) behaviour

$$K_n(x) \begin{cases} \xrightarrow{\infty} \sqrt{\frac{\pi}{2x}}\, e^{-x} \left[1 + \frac{(n+\frac{1}{2})(n-\frac{1}{2})}{2x} + \cdots\right] \\ \xrightarrow{0} \frac{2^{n-1}(n-1)!}{x^n} \left[1 - \frac{x^2}{4(n-1)} + \cdots\right] \end{cases} \tag{5.18}$$

We note these results for later use. Presently we obtain

$$Z(V,T) = \exp\left\{\frac{VT}{2\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\int_0^{\infty}\rho(m;n)\,m^2 K_2\left(\frac{nm}{T}\right)dm\right\} \qquad (5.19)$$

which shows that indeed Z is completely known if $\rho(m; n)$ is given.

5.2 Solution of the bootstrap condition

We write -- replacing E by m as integration variable --

$$Z(V,T) = \int_0^{\infty}\sigma(m,V)\,e^{-\frac{m}{T}}\,dm \qquad (5.20)$$

and require that (5.18) ≡ (5.19) and that for $V = V_0$

$$\frac{\ln\rho(m)}{\ln\sigma(m,V_0)} \xrightarrow[m\to\infty]{} 1 \qquad (5.21)$$

Now the difficulty is that the bootstrap condition implies that ρ and σ are "almost equal", and that, since $K_2(x) \sim e^{-x}$, both functions appear in an integral multiplied by exp (-m/T). Therefore, both integrals are "equal" up to some power of m, but in (5.18) the whole is exponentiated, in (5.19) not. How can these so very different expressions become equal? They can, roughly speaking, if we achieve to find a $\rho(m; n)$ which makes the integral in (5.19) behave as a logarithm, so that the exponentiation is compensated. The details are somewhat complicated. They have been discussed in the first paper on this thermodynamics [Hagedorn, 1965, henceforward quoted as (I)] and shall not be repeated here.

We can, however, see by a qualitative argument, what happens: suppose some $\rho(m)$ is given, then (5.19) can be evaluated and Z(V, T) is known. If Z(V, T) is known, then the Laplace transform (5.20) can be inverted,

and thus $\sigma(m, V_0)$ is known. Therfore, σ is a functional of ρ: for every given ρ, σ can be calculated.

$$\sigma(m, V_0) = F\left[\rho(m)\right] \tag{5.22}$$

and we may try to solve the bootstrap condition by iteration. Let us start with the worst possible assumption, namely that there is only a finite number of different types of hadrons, so that

$$\int_0^\infty \rho_B^{(0)}(m)\,dm = \beta < \infty$$

$$\int_0^\infty \rho_F^{(0)}(m)\,dm = \phi < \infty \tag{5.23}$$

The bootstrap condition $\ln \rho(m) \simeq \ln \sigma(m, V_0)$ is posed for $\bar{E} = m \to \infty$; hence we need only consider the behaviour of Z for $T \to \infty$. Then, if (5.23) is fulfilled, T can be pushed far beyond the largest masses and

$$K_2\left(\frac{nm}{T}\right) \underset{T\to\infty}{\Longrightarrow} 2\frac{T^2}{n^2 m^2} \tag{5.24}$$

This inserted into (5.19) gives

$$Z^{(0)}(V, T\to\infty) \Rightarrow \exp\left\{\frac{VT^3}{\pi^2}\sum\frac{1}{n^4}\int_0^\infty \rho(m;n)\,dm\right\}$$

$$= \exp\left\{\frac{VT^3}{\pi^2}\left(\beta\sum\frac{1}{n^4} - \phi\sum\frac{(-1)^n}{n^4}\right)\right\} \tag{5.25}$$

Using the well-known formulae for the Riemann ζ-function

$$\zeta(4) = \sum_{1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$$

$$\frac{7}{8}\zeta(4) = -\sum \frac{(-1)^n}{n^4} = \frac{7}{8}\frac{\pi^4}{90} \tag{5.26}$$

we obtain

$$Z^{(0)}(V, T \to \infty) \Rightarrow \exp\left[R\,\frac{V\pi^2 T^3}{30}\right] \tag{5.27}$$

with

$$R \equiv \int_0^\infty \rho_B(m)\,dm + \frac{7}{8}\int_0^\infty \rho_F(m)\,dm < \infty$$

The factor 7/8 is due to the Pauli principle (Fermi-Dirac statistics).

The total energy $\bar{E} = m$ is given by (3.25)

$$\bar{E} = m = T^2 \frac{\partial}{\partial T} \ln Z^{(0)} \Rightarrow \frac{R V \pi^2}{30} T^4 \tag{5.28}$$

and the energy density follows for $T \to \infty$ a Stefan-Boltzmann law:

$$\frac{\bar{E}}{V} = \varepsilon \Rightarrow \frac{\pi^2}{15} \cdot \frac{R}{2} \cdot T^4 = \frac{R}{2}\,\sigma_{St}^{em} \cdot T^4 \tag{5.29}$$

Here $\pi^2/15$ is the Stefan constant σ_{st}^{em} for ordinary electromagnetic black body radiation; indeed: for a light quantum gas $\rho_F = 0$, $\rho_B = 2\delta(m)$, $R = 2$. Thus, in passing by, we have the instructive result:

For any converging mass spectrum ρ such that

$$\int \left(\rho_B + \tfrac{7}{8}\rho_F\right) d\mu \equiv R < \infty$$

the black body radiation becomes thermodynamically similar to the electromagnetic one; its Stefan constant is $\frac{1}{2}R\,\sigma_{st}^{(em)}$.

Now we calculate $\sigma^{(0)}(\bar{E}, V)$ using the approximation (3.29) written as

$$\sigma^{(0)}(\bar{E}, V) \approx \frac{1}{\Delta E} Z(V, T) e^{\bar{E}/T} \tag{5.30}$$

Let us, in (5.28), abbreviate $RV\pi^2/30 = R'$; then $T = (\bar{E}/R')^{\frac{1}{4}}$ and $Z = \exp(1/3\, R'T^3)$. Thus

$$\sigma^{(0)}(\bar{E}, V) \approx \frac{1}{\Delta E} \exp\left(\tfrac{4}{3} R' T^3\right) = \frac{1}{\Delta E} \exp\left(\tfrac{4}{3} R'^{1/4} \bar{E}^{3/4}\right) \tag{5.31}$$

We mention, in passing by, that eq. (3.31) gives for the entropy $S \approx \ln \sigma$, thus

$$S(\bar{E}, V) = \frac{4}{3}\left(\frac{RV\pi^2}{30}\right)^{1/4} \bar{E}^{3/4} \tag{5.32}$$

and with the first law [eq. (3.14)] :

$$\frac{1}{T} = \frac{\partial S}{\partial \bar{E}} = \left(\frac{R\pi^2}{30} \cdot \frac{1}{\bar{E}}\right)^{1/4} \tag{5.33}$$

which is again the same as (5.29). Thus, for R = 2 (electromagnetic case) all formulae reduce to the well-known ones of the light quantum gas.

Identifying now $\bar{E}$ with m, we see that

$$\sigma^{(0)}(m, V_0) \sim \exp\left(\text{const.} \cdot m^{3/4}\right) \tag{5.34}$$

Our bootstrap condition requires, that $\rho(m)$ should then at least grow like exp $m^{3/4}$. Thus, our previous assumption $\rho(m)dm < \infty$ was wrong. In the next step of iteration we put

$$\rho^{(1)}(m) \sim \exp\left(\text{const } m^{3/4}\right) \tag{5.35}$$

and now the evaluation of the integrals becomes less trivial and will not be done here. The result is that

$$\sigma^{(1)}(m, V_0) \sim \exp\left(\text{const.} \cdot m / \ln m\right) \tag{5.36}$$

which grows still much faster than $\sigma^{(0)}$ and almost exponentially. But for the next iteration, $\rho^{(2)}$ must grow at least as fast as that -- and so it goes on: $\sigma^{(i)}$ grows always faster than $\rho^{(i)}$; and since we always put, for the next step, $\rho^{(i+1)} = \sigma^{(i)}$, it is obvious that each ρ grows faster than the previous one, and that we seem to approach rapidly an exponential function.

On the other hand: neither ρ nor σ can be permitted to grow faster than exponentially -- even exp (const. · m · ln m) is forbidden! -- since obviously the integrals (5.19) and (5.20) would diverge for any $T > 0$, that is: thermodynamics does not exist. One might object, that the non-existence of the partition function is no argument against $\rho(m)$ growing faster than exponentially, because one could renounce on a thermodynamic description and go back to good, old phase space. However, Frautschi (1970)

has shown, that for ρ growing *faster* than exponentially, σ grows *slower* than ρ, so that an iteration would, in that case, approach an exponential function from above*). Thus, it is clear that non-exponential solutions do not exist and that iterations lead -- from both sides -- to an exponential solution. Its existence was first proved in 1965 [see (I) for details].

That $\rho\ (m) \sim \sigma(m) \sim \exp\ (m/T_0)$ is a possible solution, can be seen qualitatively as follows: for an exponential $\rho(m)$ there will always be masses >> T and therefrom will come the main contributions to the integral in (5.19). Thus, in this case we may use [see (5.18)]

$$K_2\left(\frac{nm}{T}\right) \Longrightarrow \sqrt{\frac{\pi T}{2nm}}\, e^{-\frac{nm}{T}} \tag{5.37}$$

and this will then appear under the integral multiplied by $\exp\ (m/T_0)$. Thus, the first integral (n = 1) will diverge for $T \to T_0$, the second one (n = 2) for $T \to 2T_0$ etc. Thus, for $T \to T_0$ we consider the divergent first integral and neglect the rest of the sum (n = 2 ... ∞) which can be shown to add up to a finite constant. We put

$$\rho(m) \Longrightarrow f(m) \exp(m/T_0)$$

$$\sigma(m) \Longrightarrow g(m) \exp(m/T_0) \tag{5.38}$$

and obtain from (5.19) $\equiv$ (5.20) for $T \to T_0$ and with $\tau = (T_0 - T)/TT_0$

$$Z(V_0T) \underset{(5.19)}{\Longrightarrow} \exp\left\{ V_0 \left(\frac{T_0}{2\pi}\right)^{3/2} \int_{M_0}^{\infty} f(m)\, m^{3/2}\, e^{-m\tau}\, dm + C(M_0, V_0, T_0) \right\}$$

$$\underset{(5.20)}{\Longrightarrow} \int_0^{\infty} g(m)\, e^{-m\tau}\, dm \tag{5.39}$$

---

*) For ρ growing faster than exponentially, low-dimensional phase space with large masses is exponentially favoured against high dimensional phase space with small masses (at given total energy): resonances would not decay but coalesce; the world would be one big hadron and you would not read these lecture notes.

Here $C(M_0, V_0, T_0)$ is the finite rest of terms. Now we have two integrals, both containing the exponential function exp $(-m\tau)$, which, for $T \to T_0$ i.e. for $\tau \to 0$ tends to 1, and some non-exponential functions. One integral is exponentiated, the other not. Now, we can choose f(m), such that the integral in the exponent becomes a logarithm which compensates the exponentiation: with

$$f(m) = \frac{a_0}{m^{5/2}}$$
$$g(m) = b\, m^{\beta-1} \tag{5.40}$$

the first integral becomes

$$a_0 \int_{M_0}^{\infty} \frac{dm}{m} e^{-m\tau} = a_0 \int_{M_0\tau}^{\infty} \frac{dx}{x} e^{-x} \Rightarrow a_0 \ln\left(\frac{T_0}{M_0}\,\frac{T_0}{T_0-T}\right) \tag{5.41}$$
$$= a_0 \ln \frac{T_0}{T_0-T} + a_0 \ln \frac{T_0}{M_0}$$

and the second

$$b \int_0^{\infty} m^{\beta-1} e^{-m\tau} dm = \left(\frac{T_0}{T_0-T}\right)^{\beta} \cdot \left[b\, T_0^{\beta}\, \Gamma(\beta)\right] \tag{5.42}$$

Thus

$$Z(V_0,T) \underset{(5.19)}{\Longrightarrow} \left(\frac{T_0}{T_0-T}\right)^{\alpha_0} \exp\left[C(M_0,T_0,V_0) + \alpha_0 \ln \frac{T_0}{M_0}\right]$$

$$\underset{(5.20)}{\Longrightarrow} \left(\frac{T_0}{T_0-T}\right)^{\beta} \cdot \left[b\, T_0^{\beta}\, \Gamma(\beta)\right] \tag{5.43}$$

with

$$\alpha_0 \equiv a_0 V_0 \left(\frac{T_0}{2\pi}\right)^{3/2} \quad [\text{pure number}]$$

We now put

$$\beta = \alpha_0 = a_0 V_0 \left(\frac{T_0}{2\pi}\right)^{3/2}$$

$$b T_0^{\beta} \Gamma(\beta) = \exp\left[C(M_0, V_0, T_0) + \alpha_0 \ln \frac{T_0}{M_0}\right] \tag{5.44}$$

and have solved the bootstrap condition, because

$$\frac{\ln \rho(m)}{\ln \sigma(m, V_0)} = \frac{m/T_0 + \ln f(m)}{m/T_0 + \ln g(m, V_0)} \xrightarrow{m \to \infty} 1 \tag{5.45}$$

It should be mentioned that for sufficiently large M_0, the M_0-dependence of the argument of the exponential function in (5.44) disappears. Furthermore, (5.44) guarantees that $\sigma(m, V)$ depends exponentially on V -- as it should to ensure additivity of the entropy.