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"THERMODYNAMICS OF STRONG INTERACTIONS"

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6. DISCUSSION OF THE BOOTSTRAP SOLUTION

6.1 Our solution

Obvious as our solution of the bootstrap condition may seem -- it is not unique. Indeed: if one chooses in (5.40) a more complicated expression with a leading power m^a and $a < -5/2$, then the partition function converges for $T \to T_0$. It is then easy to reproduce in both representations (5.39) the same numerical value for $T \to T_0$ and even the same approach to this value. We could also admit $a > -5/2$, if in $\sigma(m)$ the factor $g(m)$ is allowed to behave like $\exp m^\varepsilon$ with $0 < \varepsilon < 1$, because our logarithmic bootstrap condition would still be fulfilled.

In reality, by the ansatz (5.40) we have already sharpened our logarithmic bootstrap condition to the more adequate

"week bootstrap condition"

$$\frac{c m^\alpha \rho(m)}{\sigma(m, V_0)} \underset{m \to \infty}{\Longrightarrow} 1 \quad \text{for some } \alpha, c \tag{5.46}$$

and shown, that it can be fulfilled*). We shall from now on forget the logarithmic bootstrap condition, because we got a solution of the stronger condition (5.46). As long as we do not further sharpen this condition, any $\rho(m) \sim c m^a \exp(m/T_0)$ with $a < -5/2$ will yield a solution, because then in $\sigma(m,v_0) = g(m,v_0) \exp(m/T_0)$ the function $g(m,V_0)$ is polynominally bounded. The solution $a = -5/2$ which we produced above is distinguished, however:

- $a = -5/2$ is the endpoint of the continum $a \leq -5/2$ of solutions; it corresponds to the most rapid increase of $\rho(m)$ compatible with the week bootstrap condition,
- it is the simplest one, as it causes the leading term of $g(m,v_0)$ to be a power of m.

In (I) the solution $a = -5/2$ was favoured as the most natural one. There was a philosophical argument behind it:

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*) We shall not mention the constant c any more; it might be absorbed into ρ or σ.

among all possible functions $\rho(m)$ the exponentially increasing one is the most extreme; this probably means, that strong interactions are as strong as they can be; then in $\rho(m) = f(m) \exp(m/T_0)$, $f(m)$ again should increase as fast as possible -- thus $a = -5/2$ -- and T_0 should be as small as possible -- thus of the order of m_π. Indeed $T_0 \approx 160$ MeV, which made the argument sound reasonable and suggested to take also $a = -5/2$.

Therefore in all subsequent papers, $a = -5/2$ was considered to be "the solution". Recent developments have, however, cast much doubt on this choice and brought back other solutions into serious consideration. We shall scetch this now.

6.2 The strong bootstrap condition of Frautschi

Frautschi (1970) observes, that in the thermodynamic treatment $\sigma(m,V_0)$ contains still the states corresponding to the motion of the fireball as a whole and secondly, that in our expression (5.19) zero- and one-particle states are contained, which in a true bootstrap of hadrons should be excluded. He prefers then to work in phase space with full momentum conservation, starting with two-particle states. As the conservation of total momentum will decrease $\sigma(m,V_0)$ asymptotically by a factor $m^{-3/2}$ (see the comments following the bootstrap reasoning in Section 4.5) one might indeed hope, that a strong bootstrap condition $\rho(m) \to m^{-3/2}\sigma(m,V_0)$ could be satisfied.

Frautschi uses $\rho(m)$ as "input" function and calls it $\rho_{in}(m)$, while the total density of states *with* momentum conservation is the "output" function called $\rho_{out}(m)$ [thus $\rho_{out}(m) \to m^{-3/2}\sigma(m,V_0)$].

His equation then reads [compare our Eqs. (4.1), (4.6), (4.7), (4.8), (4.10)]

$$\rho_{out}(m) = \sum_{n=2}^{\infty} \left[\frac{V_0}{(2\pi)^3}\right]^{n-1} \frac{1}{n!} \prod_{i=1}^{n} \int dm_i d^3p_i \, \rho_{in}(m_i) \, \delta\Big(m - \sum_i \sqrt{p_i^2 + m_i^2}\Big) \, \delta^3\Big(\sum_i \vec{p}_i\Big) \qquad (5.47)$$

and his "strong bootstrap condition" is

$$\frac{\rho_{in}(m)}{\rho_{out}(m)} \underset{m\to\infty}{\Longrightarrow} 1 \qquad (5.48)$$

Which is the weak condition (5.46) sharpened by requiring $\alpha = 3/2$. He shows that this strong condition can only be satisfied by

$$\rho_{in}(m) \to c m^a e^{m/T_0}$$

$$a < -5/2$$

The difference with our result is that his power of m must be *smaller* than $-5/2$ while we found that $-5/2$ is also allowed. There is no contradiction; we differ by physical assumptions:

Frautschi's $\rho_{out}(m)$ counts all possible compound states (in our words: fireballs) made up by any possible combination of hadrons, whose mass spectrum is $\rho_{in}(m)$. Requiring $\rho_{out} \to \rho_{in}$ is equivalent to the assumption that for $m \to \infty$ there are no constraints on $\rho_{in}(m)$ which are not also imposed on $\rho_{out}(m)$. These latter ones are explicitly written down in (5.46): energy- and momentum conservation, that is all. It is, however, conceivable, that there are some more constraints on $\rho_{in}(m)$, which have been forgotten in (5.46).

One constraint, not made explicit in (5.46), is the Pauli exclusion principle. This will probably not matter, since in our formulation its neglect will only have an influence on the terms $n \geq 2$ in Eq. (5.19), and these terms are regular at $T = T_0$ and did not enter our discussion of what the power of m in front of the exp m/T_0 of $\rho(m)$ should be.

Another constraint may be more important: we had to admit that ρ and σ might differ by $m^{3/2}$, because we neglected momentum conservation in computing Z. Frautschi does not neglect this; therefore he need no longer admit a factor $m^{3/2}$ between ρ_{in} and $\rho_{out} \approx m^{-3/2}\sigma$. However, angular momentum conservation, if introduced into (5.46) would decrease ρ_{out} by some further power (see the literature quoted at the end of Section 3.5). Suppose now, that the states counted in the hadronic mass spectrum $\rho(m) \equiv \rho_{in}(m)$ obey some regularity with respect to their total spins. This regularity has then be neglected in (5.46) and thus ρ_{out} does not

obey it; hence ρ_{out} and ρ_{in} could again differ by a function of m. In fact to the ρ_{out} of (5.46) belongs a definite statistical distribution of spin values, while the hadronic mass spectrum ρ_{in} might have another one. This would depend very much on the structure of the set of all Regge trajectories and is presently not well enough understood to exclude that ρ_{in} and ρ_{out} could still differ by a function of m which does not tend to 1 for $m \to \infty$.

A last difference might come from constraints on other quantum numbers, e.g. it could be that in ρ_{in} exotic states are absent, while they are included in ρ_{out}. This again can possibly lead to a difference between these two functions, which does not disappear for $m \to \infty$.

In all these cases, Frautschi's strong bootstrap condition might be modified with the result that the power a in $\rho(m) \to cm^a \exp(m/T_0)$ is no longer restricted to $a < 5/2$ but that $a = -5/2$ becomes also allowed -- it needs very little; but *how* little depends on physics which we do not yet understand well enough. To sum up:

> Our *weak* bootstrap condition $m^\alpha \rho(m) \to \sigma(m)$ -- α not fixed -- does not touch on presently unknown details and allows any $\rho(m) \to cm^a \exp(m/T_0)$ with $a \leq -5/2$;
>
> any *strong* bootstrap condition -- α fixed -- contains implicit or explicit assumptions about the constraints imposed on the hadronic mass spectrum $\rho(m) \equiv \rho_{in}(m)$.

Frautschi's version just excludes the endpoint $-5/2$ of the interval of allowed powers of m. On can go one step further. Nahm (1971) considers the thermodynamical formulation with momentum conservation [which essentially means replacing $\sigma(m)$ by $\rho_{out} \times m^{3/2}$]. Imposing then Frautschi's bootstrap condition, he writes our Eq. (5.39) in the equivalent form

$$Z(V_0, T \to T_0) \Longrightarrow \exp\left\{A(\tau) \int_{M_0}^{\infty} \mu^{a+\frac{3}{2}} e^{-\mu\tau} d\mu + B(\tau)\right\}$$
$$\Longrightarrow C(\tau) \int_{M_1}^{\infty} \mu^{a+\frac{3}{2}} e^{-\mu\tau} d\mu + D(\tau) \qquad (5.49)$$

The strong bootstrap condition is here already built in; the functions $A(\tau)$, $B(\tau)$, $C(\tau)$ and $D(\tau)$ need not be specified; important is only that they all are regular at $\tau = 0$, i.e. can be expanded in a power series.

Now, with $m\tau = x$ we have for the integrals

$$\int_M^\infty m^{a+\frac{3}{2}} e^{-m\tau}\, dm = \left(\frac{1}{\tau}\right)^{a+\frac{5}{2}} \int_{M\tau}^\infty x^{a+\frac{3}{2}} e^{-x}\, dx$$

$$= \left(\frac{1}{\tau}\right)^{a+\frac{5}{2}} \left[\Gamma\left(a+\tfrac{5}{2}\right) + \cdots\right] \tag{5.50}$$

We put

$$a + \frac{5}{2} = \gamma \tag{5.51}$$

then (5.49) becomes

$$Z(V_0, T \to T_0) \Longrightarrow \exp\left\{A_1(\tau)\left(\frac{1}{\tau}\right)^\gamma + B(\tau)\right\}$$

$$\Longrightarrow C_1(\tau)\left(\frac{1}{\tau}\right)^\gamma + D(\tau) \tag{5.52}$$

where $\Gamma(\gamma) + \ldots$ is absorbed into $A_1(\tau)$ and $C_1(\tau)$. In (I) we argued that $\gamma > 0$ is excluded, because the upper line would diverge exponentially for $\tau \to 0$ and the lower not. For $\gamma = 0$ -- which just is our old solution $a = -5/2$ of the *weak* bootstrap -- we have in (5.50) to take the limit $a \to -5/2$ (from below) with the result $\sim \ln 1/\tau$. This is not acceptable as a solution in the present *strong* bootstrap case, because now the upper line of (5.52) gives $\to \tau^{-A_1}$ while the lower yields $C_1 \ln 1/\tau$. Hence $\gamma \geq 0$ is excluded. Let now

$$\gamma \equiv -\varepsilon \quad ; \quad \varepsilon > 0 \tag{5.53}$$

We then find

$$Z(v_0, T \to T_0) \Longrightarrow \exp\left\{A_1(\tau)\tau^{\varepsilon} + B(\tau)\right\}$$

$$\Rightarrow \; C_1(\tau)\tau^{\varepsilon} + D(\tau) \tag{5.54}$$

Expanding the exponential function yields

$$\left[1 + A_1(\tau)\tau^{\varepsilon} + \tfrac{1}{2}A_1^2(\tau)\tau^{2\varepsilon} + \cdots\right] \times \exp B(\tau)$$

$$\Rightarrow \; C_1(\tau)\tau^{\varepsilon} + D(\tau) \tag{5.55}$$

The lower line contains only τ^{ε}, the upper τ^{ε}, $\tau^{2\varepsilon}$ ...., thus ε must be half-integral; therefore $a = \gamma - 5/2 = -\varepsilon - 5/2$ is limited to

$$a = -3, -4, -5, \ldots. \tag{5.56}$$

because the values $a = -7/2, -9/2, \ldots$ would again produce logarithmic expressions which cannot be matched. This argument can be further sharpened by more close inspection of the strong bootstrap condition; the result is that only $a = -3$ is a solution. We thus state Nahm's result (1971)

> if the strong bootstrap condition $\rho_{in} \to \rho_{out}\ [\rho(m) \to m^{-3/2}\sigma(m, v_0)]$ is posed, only one solution

$$\rho(m) \Longrightarrow \frac{c}{m^3} e^{m/T_0} \tag{5.57}$$

remains; the constants c and T_0 are related to V_0.

We repeat, that the strong bootstrap condition may be too strong -- just a little too strong, so that our solution $\rho(m) = cm^{-5/2} \exp(m/T_0)$ may become again allowed.

6.3 The mass spectrum in the Veneziano model

In this part I will be very short, because the relation between the thermodynamical and the dual approach is not yet understood. The central result of our thermodynamics is *that a complete and general hadronic bootstrap is soluble and that the hadronic man spectrum is necessarily exponential.* Exactly this was also the outcome of the attempt to "factorize" the Veneziano n-point function; factorization is another technical approach to bootstrap. The surprise was great, when Fubini and Veneziano (1969) and Fubini, Gordon and Veneziano (1969) found this result. In the meantime it became considered one of the most significant implications of dual bootstrap models (instead of a serious disease). To this attitude the result of the thermodynamical bootstrap and some other general arguments [Brout 1969, Krzywicki 1969] based on duality have certainly contributed.

The present situation seems to be the following one [Lee and Leung 1970]: the Veneziano model predicts [Huang and Weinberg 1970]

$$\rho^V(m) \underset{m\to\infty}{\Longrightarrow} \sqrt{2\alpha'} \left(\frac{D}{24\alpha'}\right)^{\frac{D+1}{4}} \cdot \frac{m}{(m_0^2+m^2)^{\frac{D+3}{4}}} \exp\left[2\pi\sqrt{\frac{D\alpha'}{6}(m^2+m_0^2)}\right] \tag{5.58}$$

where the constants mean

$$\left.\begin{aligned} \alpha_0 &= \text{intercept} \\ \alpha' &= \text{slope} \end{aligned}\right\} \text{ of leading Regge trajectory}$$
$$m_0 = \sqrt{\alpha_0/\alpha'} \approx 1 \text{ GeV} \tag{5.59}$$
$$D = \text{dimension of oscillator-operator}$$

The "dimension" D is at least 4; it probably is 5, less probably larger than 5. We cannot go into these questions here, it may suffice to say that presently D = 5 is favoured [Lovelace 1969].

Let us push m so far that $m \gg m_0$. Then

$$\rho^V(m) \underset{m\to\infty}{\Longrightarrow} \text{const.}\; m^{-\frac{D+1}{2}}\, e^{m/T_0}$$

$$T_0 = \sqrt{\frac{3}{2\pi^2 D\alpha'}} \tag{5.60}$$

This formula is surprisingly near to what we found as solutions to the weak or strong bootstrap conditions: for D = 4, 5, 6 ... we obtain the power of m to be $-5/2$, -3, $-7/2$ .... and just $-5/2$ and -3 were favoured by us in the weak and strong case respectively. The Veneziano model seems not to require $D \geq 6$ and we do not like to accept a power $\leq -7/2$ for reasons which will be discussed below -- a satisfactory coincidence.

Furthermore, with $\alpha' \cong 1\ \text{GeV}^{-2}$ and D = 4, 5, 6 one obtains

$$T_0{}^V = \begin{cases} 196\ \text{MeV} & D = 4 \\ 174\ \text{MeV} & D = 5 \\ 146\ \text{MeV} & D = 6 \end{cases} \tag{5.61}$$

which all are not far from the experimental value $\approx$ 160 MeV (see below).

The structures of the Veneziano n-point function with factorization and of our thermodynamics are technically very different, but similar in philosophy: both are dual and both postulate a general and universal bootstrap. In the Veneziano case fermious are not yet incorporated, but the thermodynamic approach shows that this does not matter so much (if there were only bosons, the terms $n \geq 2$ in (5.19) would be affected - but they are regular at T_0 and do not enter the discussion of the solution). Thus it seems that

a general and universal hadronic bootstrap leads to an exponential mass spectrum

rather independently of the structure of the formalism.

6.4 Comparison with the experiment, physical consequences

So far we have discussed the mathematical situation of the bootstrap solution, the exponential spectrum. We shall now compare the prediction

$$\rho(m) \Longrightarrow c\, m^a \exp(m/T_0) \tag{5.62}$$

with the experiment. So far the spectrum of particles and resonances is known only up to about 2-3 GeV and only below 1 GeV we may assume to have a complete knowledge. The asymptotic formula (5.62) diverges for $m \to 0$ and must be modified. One possibility is to take the formula of Huang and Weinberg (5.58) and try to fit it to the experimental mass distribution. Long before the Huang-Weinberg formula was derived, L. Fogli proposed to continue our solution with $a = -5/2$ to low masses by

$$\rho(m) \approx \frac{a_0}{(m^2+m_0^2)^{5/4}} \exp(m/T_0) \tag{5.63}$$

Fig. 7 shows, how well one can fit the mass spectrum with such a formula.

However, reasonable fits are possible with the Huang-Weinberg formula as well (e.g. $D = 6$, $\alpha' = 1.2$ GeV^{-2}, $\alpha_0 = 0.8$ is not bad, it gives $T_0 = 146$ MeV; with $D = 5$ a good fit can be achieved only at the cost of a much too high slope $\alpha' = 1.8$ and rather low intercept $\alpha_0 = 0.4$ leading to $T_0 = 130$ MeV) or with another guess-formula

$$\rho(m) \approx \frac{a_0}{(m_0+m)^{5/2}} \exp(m/T_0) \tag{5.64}$$

with $a_0 = 1.47 \times 10^5$ MeV$^{3/2}$

$m_0 = 1000$ MeV

$T_0 = 168.6$ MeV

This last is actually the best fit obtained so far, but in drawing hardly distinguishable from the fit of figure 7.

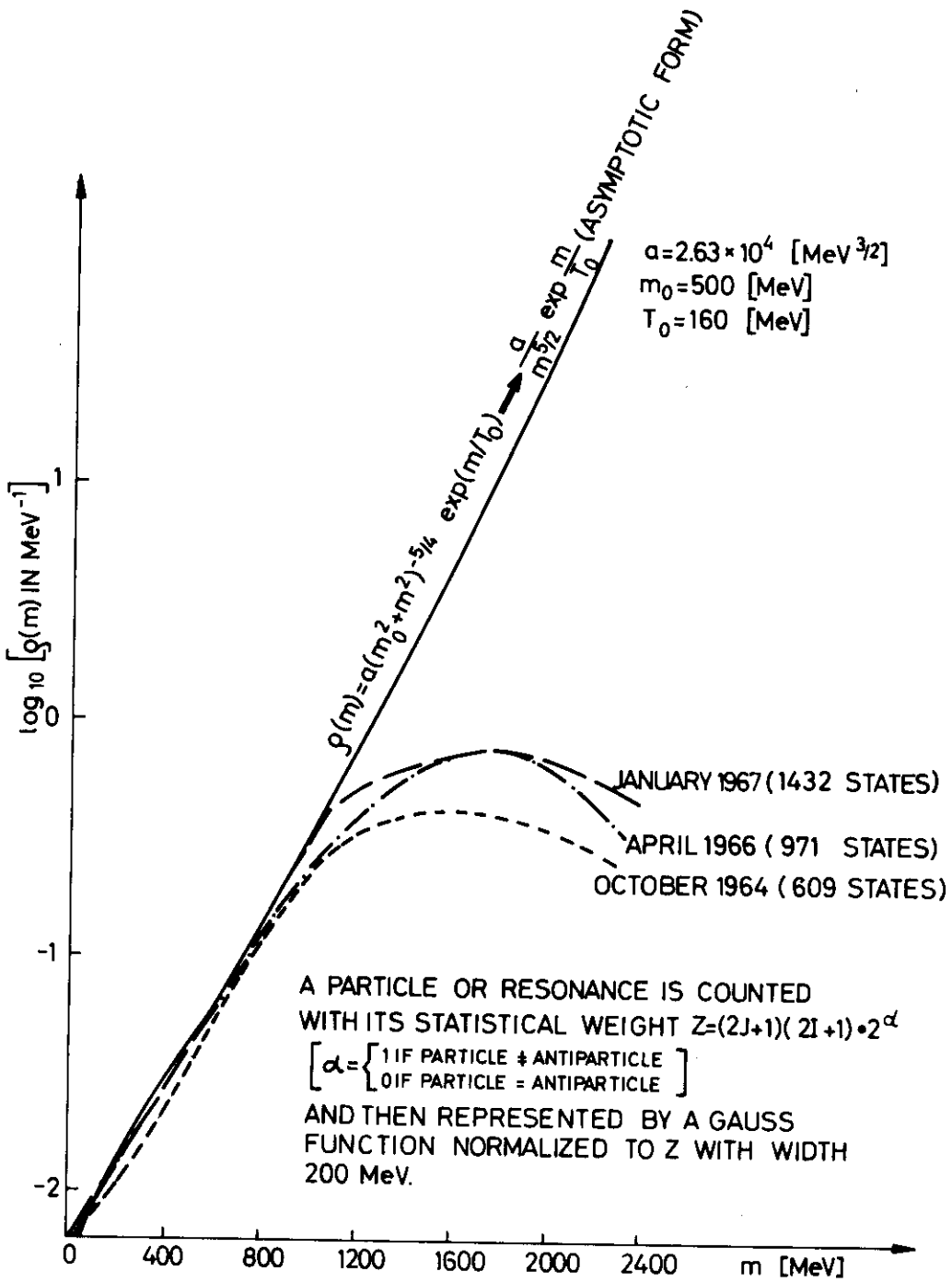

Fig. 7 The smoothed experimental mass spectrum as it developed from 1964 to 1967 and the function

$$\rho(m) = a(m_0^2 + m^2)^{-5/4} \exp(m/T_0)$$

which has the asymptotic form required by the theory. The figure is taken from (II).

The lesson we learn from these different fits is, that the well-known parts of the spectrum do not yet suffice to fix T_0 and other constants or even the power of m; reasonable to good fits can be obtained with

$$130\,\text{MeV} \lesssim T_0 \lesssim 180\,\text{MeV} \tag{5.65}$$

and the value $T_0 = 160$ MeV, adopted in all previous papers, might be a good choice. *What is important, is that the known part of the mass spectrum does not contradict the exponential spectrum.*

What are the physical consequences? The most striking one is, that

T_0 is the universal maximum temperature for all matter

because above T_0 the partition function and all thermodynamic functions do not exist. It is a universal maximum temperature, because any matter (e.g. neutrinos) brought to some temperature $T > T_0$ would, in the long run, by their mutual collisions create hadrons and then further hadron creation would rapidly use up the available kinetic energy until $T \lesssim T_0$ is reached. That there exists a highest temperature, is not so strange as it first seems: for an ideal gas with fixed number N of particles, the temperature increases like $^2/_3$ E/N (see Section 3.6). If particle creation is permitted, then the temperature increases much more slowly, namely like $(E/V)^{1/4}$ [see Eq. (5.29)], provided the spectrum is finite. If an exponential spectrum is permitted, then, for large E/V, the temperature grows very slowly and approaches a limit T_0. It does so, because with an exponential mass spectrum it is much easier to create new particles than to increase the kinetic energies of those present. The whole has a striking similarity to boiling and therefore I once called T_0 the boiling point of hadronic matter (Hagedorn 1968a).

Now if T_0 is the maximum temperature, then the momentum distribution of particles emitted from highly excited fireballs, must tend to a limiting distribution, namely the Planck-distribution (4.9) with $T(E) = T_0$. Since in real collisions the longitudinal momentum distribution is largely determined by the collective motions [see Eq. (2.9)], we have to look for the transverse momentum distribution, which is invariant under the Lorentz-transformations in Eq. (2.9). This transverse momentum

distribution should be again of a similar type as the Planck distribution. This is in fact, what one sees experimentally. What is important, however, is that for very large collision energies these distributions are indeed characterized by $T(E) \to T_0$ with the *same* $T_0 \approx 160$ MeV as in the mass spectrum:

- exponential mass spectrum $\rho(m) \sim \exp(m/T_0)$,
- maximum temperature T_0 and
- limited transverse momenta --

these three are in reality only different aspects of one and the same fundamential fact, namely of the general and universal hadron bootstrap.

6.5 Thermodynamic functions

After the bootstrap is solved and the asymptotic mass spectrum has been extrapolated down to the region where we know it experimentally, we can evaluate the partition function as given by Eq. (5.19) numerically. It turns out, that for $m \gtrsim 1$ GeV the argument nm/T of the Hankel function K_2 becomes so large -- because $T < T_0 \approx 160$ MeV -- that all terms with $n \geq 2$ can be neglected there. For small T this is obvious, for $T \to T_0$ it is still a very good approximation even if we choose the solution where $\rho(m) \to m^{-3} \exp(m/T_0)$; if we choose the one with $\rho(m) \to m^{-5/2} \exp(m/T_0)$, then the first term ($n = 1$) diverges and the rest can be neglected anyway. Thus, the procedure is to split the integral in one from m_π to $M \approx 1$ GeV, use there the experimental mass spectrum $\rho(m;n) = \rho_B(M) - (-1)^n \rho_F(m)$ and evaluate the sum over as many terms $n = 1, 2, 3 \ldots N$ as one needs for a given accuracy and then to integrate from $M \approx 1$ GeV to ∞ with $\rho(m)$ given by (5.64) or some similar extrapolation, neglecting all terms $n \geq 2$. This has been carried out explicitly in (II) and a good numerical approximation to the result has been given in another paper (Hagedorn 1970a Section 4.1) both assuming the solution $\rho(m) \to cm^{-5/2} \exp(m/T_0)$. Instead of going into any detail, we only remark that

> after having choosen one of the bootstrap solutions $\rho(m) \to cm^a \exp(m/T_0)$ and extrapolating it down to the known mass spectrum, the whole $\rho(m;n)$ is known and the partition function $Z(V,T)$ is a known function of V (any V, not restricted to V_0) and T. As

all thermodynamic functions: energy, entropy, pressure etc., as well as all distribution functions are derivable from Z(V,T), the whole problem of a thermodynamic description of highly excited strongly interacting hadronic matter is solved (up to the ambignity of which bootstrap solution should be taken).

The asymptotic behaviour for $T \to T_0$ does, of course, not depend on the detailed structure of $\rho(m)$ at low masses, here only the asymptotic $\rho(m)$ matters.

One should think that in view of the exponential factor the power a of $\rho(m) \to cm^a \exp(m/T_0)$ does not matter at all -- this is not so. Let us consider the two cases $a = -5/2$ and $a = -3$. The first one has been worked out when we tried to find a solution of the bootstrap: Eq. (5.43) gave

$$\ln Z_{-5/2}(V_0, T \to T_0) \Longrightarrow V_0 C(M_0, T_0) + \\ + \alpha_0 \ln \frac{T_0}{M_0} + \alpha_0 \ln \left(\frac{T_0}{T_0 - T}\right) \tag{5.66}$$

where with a_0 and T_0 taken from the fit of Fig. 7 we find

$$\alpha_0 = a_0 V_0 \left(\frac{T_0}{2\pi}\right)^{3/2} = 5{,}37 \quad (\text{pure number}) \tag{5.67}$$

On the other hand, with $a = -3$, the partition function converges at $T \to T_0$ to a constant, because now (5.41) becomes with $\rho(m) \to cm^{-3} \exp(m/T_0)$

$$c \int_{M_0}^{\infty} \frac{dm}{m^{3/2}} e^{-m\tau} = c\sqrt{\tau} \int_{M_0\tau}^{\infty} dx\, x^{-3/2} e^{-x} \approx c\sqrt{\tau}\, \Gamma(-\tfrac{1}{2})$$

hence now, with $\Gamma(-\frac{1}{2}) < 0$ and $\tau = (T_0 - T)/TT_0$

$$\ln Z_{-3}(V_0, T \to T_0) \Longrightarrow -A\sqrt{T_0 - T} + B \; ; \; A > 0 \tag{5.68}$$

so that ln Z_{-3} approaches the limit B from below with infinite slope. I have not worked out the values of A and B, but both are proportional to V_0 as were C and α_0 in the $a = -5/2$ solution. The pressure is given by (3.25) as

$$P = T \frac{\partial}{\partial V} \ln Z(V,T) \tag{5.70}$$

Thus (note that for black body radiation P is a function of T alone!)

$$\begin{aligned} P_{-5/2}(T \to T_0) &\Longrightarrow A_1 + A_2 \ln\left(\frac{T_0}{T_0 - T}\right) \\ P_{-3}(T \to T_0) &\Longrightarrow \frac{B}{V_0} - \frac{A}{V_0}\sqrt{T_0 - T} \end{aligned} \tag{5.71}$$

Thus for $a = -5/2$ ln Z and P diverge logarithmically for $T \to T_0$, while for $a = -3$ both converge to constant values, however approaching them with infinite slope (analogy to boiling!).

The energy diverges in both cases; Eq. (3.25):

$$E = T^2 \frac{\partial}{\partial T} \ln Z(V,T) \tag{5.72}$$

$$\begin{aligned} E_{-5/2}(V_0, T \to T_0) &\Longrightarrow T_0 \alpha_0 \frac{T_0}{T_0 - T} \\ E_{-3}(V_0, T \to T_0) &\Longrightarrow \frac{A}{2} T_0^2 \frac{1}{\sqrt{T_0 - T}} \end{aligned} \tag{5.73}$$

These examples may suffice to show that the physical differences of the two favoured solutions $a = -5/2, -3$ are not completely negligible. We shall encounter others.

Here is also the place to show why we would not like to accept $\rho(m) \to cm^a \exp(m/T_0)$ with $a \leq -7/2$. Namely, for $a < -7/2$ we obtain for the leading integral in ln Z (Eq. (5.41).

$$c\int_{M_0}^{\infty}\frac{dm}{m^{\alpha}}e^{-m\tau} \cong c\,\tau^{\alpha-1}\,\Gamma(-\alpha+1)\,;\quad \alpha>2 \tag{5.74}$$

Thus

$$\ln Z \Rightarrow -A(T_0-T)^{\beta}+B\,;\quad \beta>1 \tag{5.75}$$

and

$$E=T^2\frac{\partial}{\partial T}\ln Z \Rightarrow \beta A T_0^{\,2}(T_0-T)^{\beta-1}+C \tag{5.76}$$

We see that now for $T \to T_0$ also $E \to$ const. That is: above a certain energy E_0 no thermodynamics is possible if in $\rho(m)$ the power of m is smaller than $-7/2$; actually the case $a = -7/2$ leads to the same behaviour, E_0 is then approached with infinite slope. As long as the solutions $a = -5/2$ and $a = -3$ are acceptable, we may discard $a \leq -7/2$ as unphysical (see also Huang and Weinberg, 1970)

Conclusion

The question: which power a of m stands in front of the exponential function in the hadronic mass spectrum is all but trivial. Its answer depends on presently unknown but important details of strong interaction physics and it influences non-negligibly the asymptotic behaviour for $E \to \infty$ not only in particle physics but also in cosmic physics (Hagedorn 1970a, Huang and Weinberg 1970).

Good numerical approximations of all thermodynamic functions have been given for the solution $\rho(m) \to cm^{-5/2}\exp(m/T_0)$ in the paper (Hagedorn 1970a). There also the analytical expressions for all charac-

teristic quantities of a hypothetical fireball consisting only of particles with mass $m \gg T_0$ have been derived (for $T \to T_0$ real fireballs approach this hypothetical fireball!).